Instability and Mesh Dependence
Part I – Theory or “what the hell is ECRIT?”

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A day in the life of a consultant

Paul, I am creating a GISSMO card, what do I put for ECRIT?

euhh….
well…..
In the uniaxial case it is the plastic strain at necking…

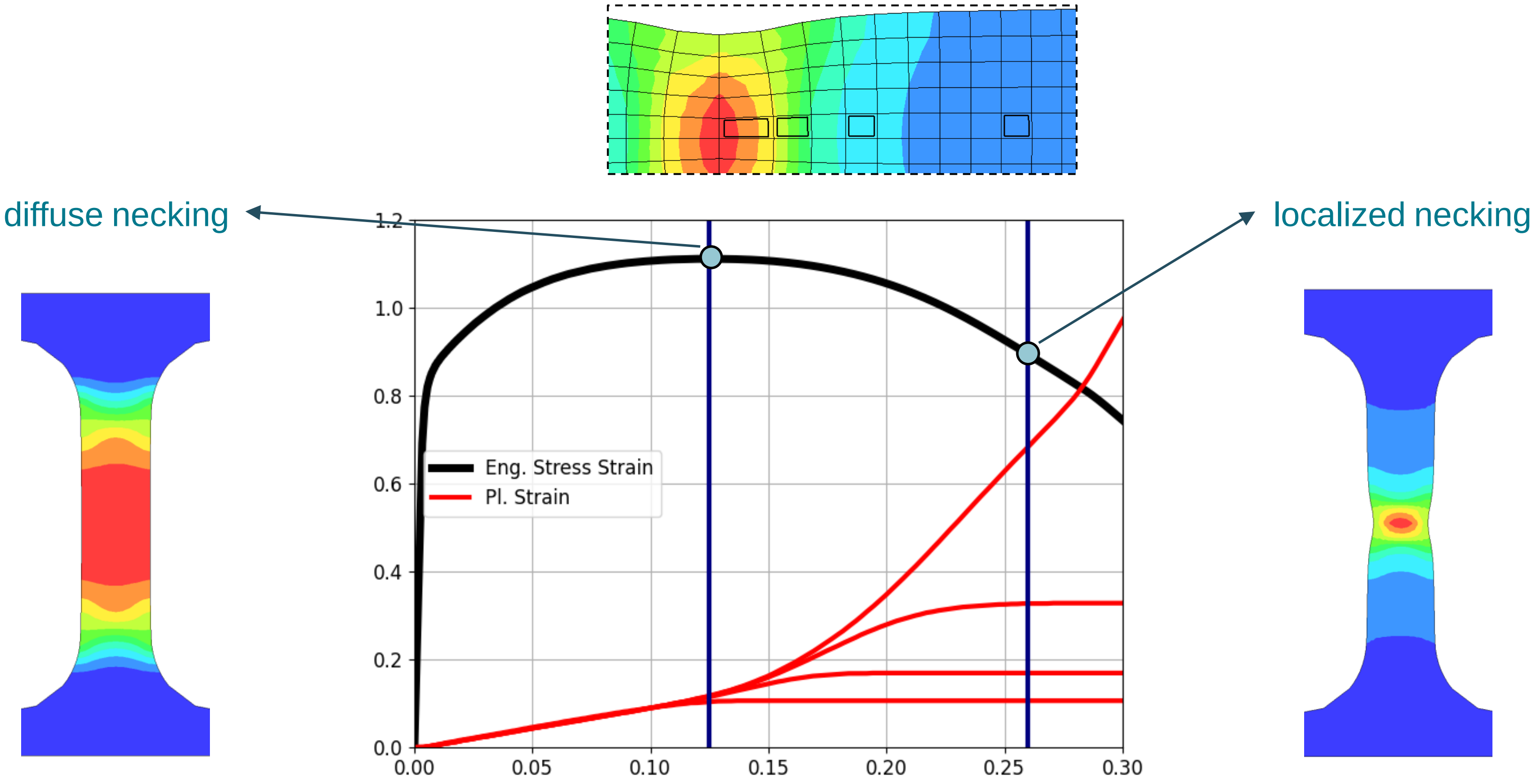
GISSMO and ECRIT

- GISSMO requires the definition of an ECRIT curve (shells) or an ECRIT surface (solids) to define the 'start of instability'
- For us that always meant the 'start of mesh dependency'
- The point on the ECRIT curve corresponding to uniaxial tension indicates the start of diffuse necking and this point is very well known, under general plane stress or full 3D loading conditions this is not the case
- Intuitively, ECRIT is the locus where
  - we lose the homogeneous state of stress/strain
  - localisation of plastic strain starts
  - structural instability is detected
  - spurious mesh dependency is detected in the numerical simulations
- In the case of uniaxial tension all these events are known to coincide at the moment of maximum force

Diffuse and localized necking

Uniaxial tension

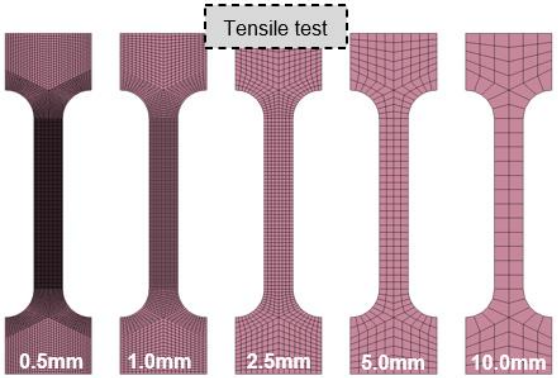
- Start of diffuse necking
- Loss of homogeneous state of stress/strain
- Start of structural instability
- Start of localization of plastic strain
- Moment of maximum force
- Start of mesh dependency

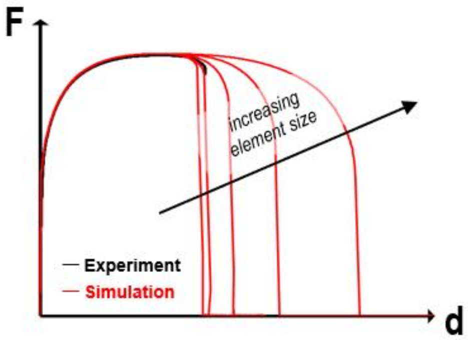


What we've known for a long time

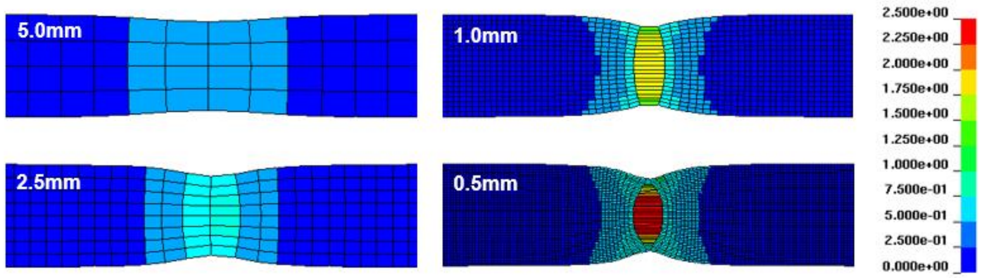
Spurious mesh dependence observed in simulations

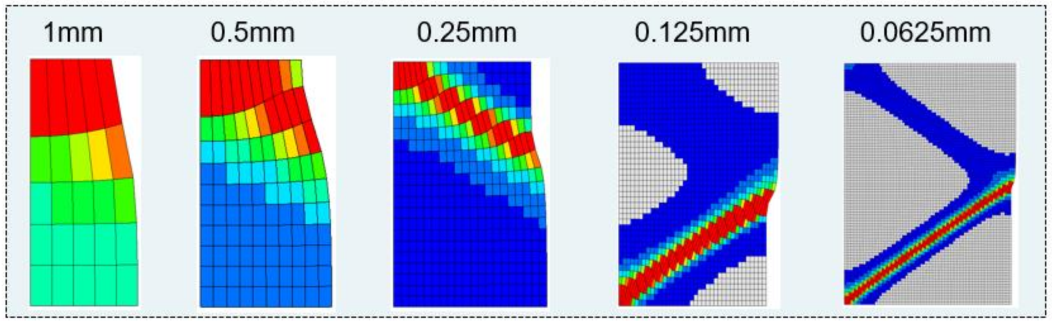
- Start of diffuse necking
- Loss of homogeneous state of stress/strain
- Start of structural instability
- Start of localization of plastic strain
- Moment of maximum force
- Start of mesh dependency





Significant differences in local plastic strain at the same global displacement





Concentration in a single element band is within the local continuum theory eventually unavoidable

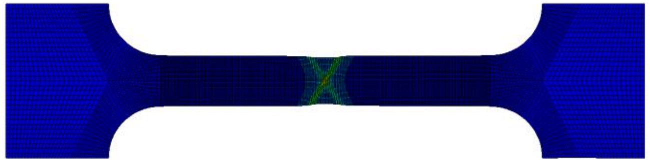
All kinds of structural instabilities

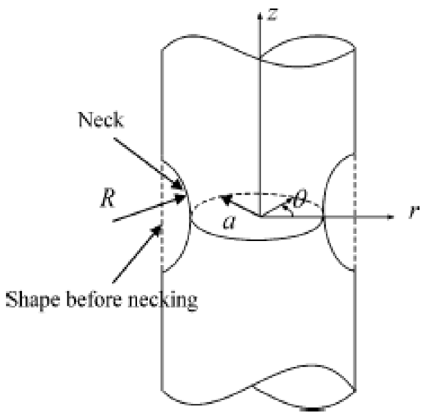
Only diffuse necking happens in both 2D and 3D states of stress

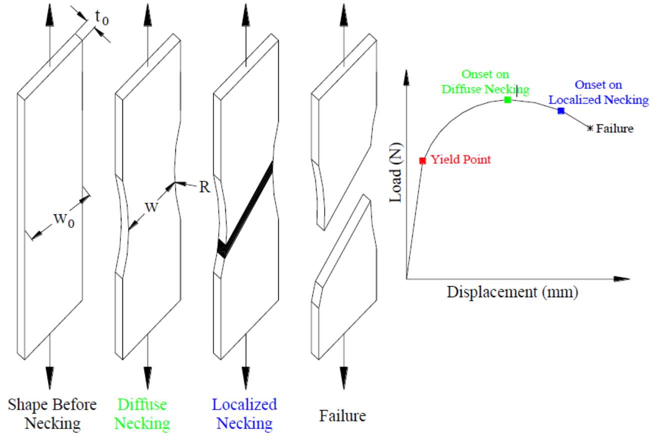
In general, two effects may be distinguished in **uniaxial tension in thin metal sheets**:

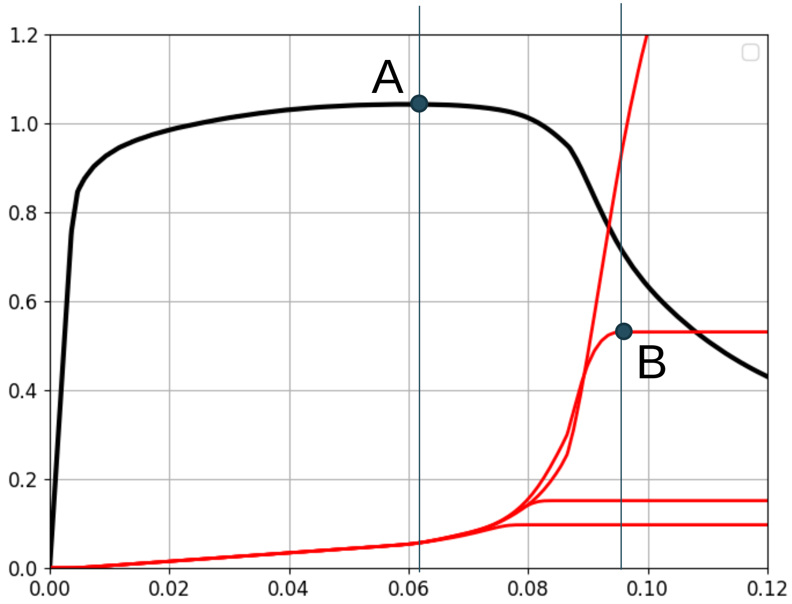
A) Diffuse necking:
Onset of formation of non-uniform strains. Usually, this is the point of maximum load in the force-displacement diagram.

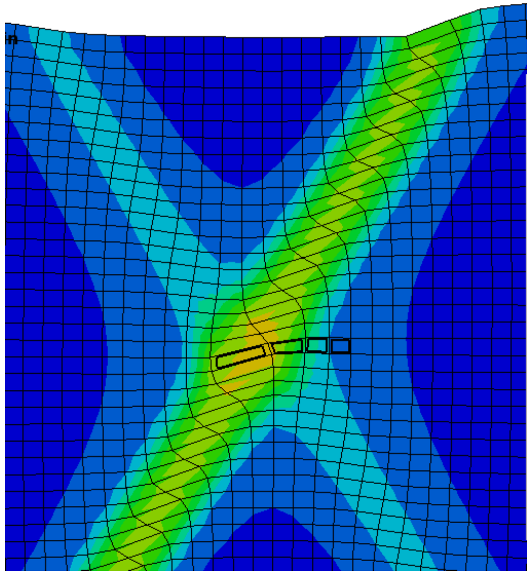
B) Localized necking:
This is understood as the point when strain localizes a narrow band of constant width causing pronounced thinning and leading the coupon to failure











2D and 3D instability criteria

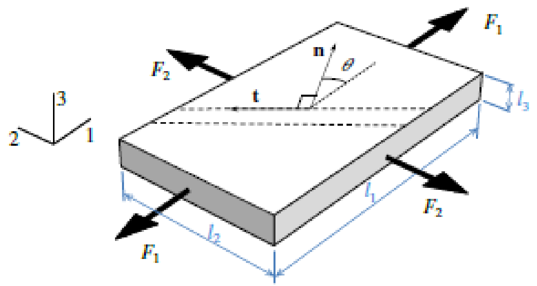
State of the art (Abed-Meraim et al., 2014, Bouktir, 2017)

Engineering criteria (1D-2D)

	Necking type	Remark
Hill'52	Localized	Ratio of strain rates inside/outside loc. band
Hora	Localized	Maximum Force of major load component
Considère	Diffuse 1D	Maximum Force in uniaxial case
Swift	Diffuse 2D	Maximum Force in two directions
MK	Localized	Imperfection criterion
HN	Localized	Imperfection criterion
Keeler	Localized	Empirical

Loss of uniqueness and bifurcation criteria (2D-3D)

	Necking type	Remark
General Bifurcation	Diffuse	Second order work
Limit-Point Bifurcation	Diffuse	Stationary stress state
Loss of Strong Ellipticity	Localized	Discontinuity; Symm. Acoustic tensor
Loss of Ellipticity	Localized	Discontinuity, Acoustic tensor



The concept… Koblenz, May 15th 2019

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- Introduction and Motivation
- General (GBC) and Limit Point Bifurcation (LPBC) Criteria
- Evaluation of the bifurcation criteria for J2 plasticity

Part II

- Validation of GBC and LPBC for J2 elastoplasticity by numerical simulations
- Conclusions

Like many other things it starts with Sir Rodney Hill (11/6/1921-2/2/2011)

- The uniqueness criterion of Sir Rodney was based on the stiffness matrix which couples the rate of the nominal stress to the transposed of the velocity gradient
- This is a global criterion and the loss of uniqueness is a function of the material law, geometry and boundary conditions… we should always keep this in mind
- Implementation at the level of the integration point will force us to use a local version of the uniqueness criterion (no integral)

Journal of the Mechanics and Physics of Solids, 1958, Vol. 6, pp. 236 to 249. Pergamon Press Ltd., London

A GENERAL THEORY OF UNIQUENESS AND STABILITY IN ELASTIC–PLASTIC SOLIDS

By R. Hill
Department of Mathematics, University of Nottingham

(*Received 27th February, 1958*)

$$\int \Delta \dot{s}_{ij}\, \Delta \left(\frac{\partial v_j}{\partial x_i}\right) dV = 0.$$

A sufficient condition for uniqueness is therefore that

$$\int \Delta \left\{\frac{\partial E}{\partial (\partial v_j/\partial x_i)}\right\} \Delta \left(\frac{\partial v_j}{\partial x_i}\right) dV > 0 \qquad (20)$$

$$\int c_{ijkl}\, \xi_{ij}\, \xi_{kl}\, dV > 0$$

for arbitrary ξ_{ij} and for

$$c_{ijkl} = \frac{\partial^2 E}{\partial (\partial v_j/\partial x_i)\, \partial (\partial v_l/\partial x_k)}$$

corresponding to *any* velocity field.

$$\xi_{ij} = \Delta (\partial v_j/\partial x_i).$$

Molding the material law a bit

- We start from the usual incremental elasto-plastic material law with the Jaumann rate:

$$\mathbf{L} = \frac{\partial \dot{\mathbf{x}}}{\partial \mathbf{x}} \qquad \mathbf{D} = \frac{1}{2}\left[\mathbf{L} + \mathbf{L}^T\right] = \mathbf{D}^T \qquad \mathbf{W} = \frac{1}{2}\left[\mathbf{L} - \mathbf{L}^T\right] = -\mathbf{W}^T \qquad \dot{\boldsymbol{\sigma}} = \overset{4}{\mathbf{C}}_{ep} : \mathbf{D} + \mathbf{W}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{W}$$

- And recall the definition of the 1PK stress and its transposed called the nominal stress:

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0} \qquad j = |\mathbf{F}| \qquad \mathbf{P} = j\boldsymbol{\sigma}\mathbf{F}^{-T} \qquad \mathbf{N} = j\mathbf{F}^{-1}\boldsymbol{\sigma} = \mathbf{P}^T$$

- In Hill 1958 the material law is reformulated in terms of the rate of the nominal stress and the transposed of the velocity gradient, **D** is a 4th order stiffness tensor:

$$\dot{\mathbf{N}} = \overset{4}{\mathbf{D}}_{NLT} : \mathbf{L}^T \qquad \dot{N}_{ij} = D_{ijkl} L_{kl}$$

- But there are also other possibilities that one finds in the literature

4 variants of the material law

- At least 4 variants are found in modern literature:

$$\dot{\mathbf{P}} = \overset{4}{\mathbf{D}}_{PL} : \mathbf{L} \qquad \dot{P}_{ij} = \left(D_{PL}\right)_{ijkl} L_{kl}$$

$$\dot{\mathbf{P}} = \overset{4}{\mathbf{D}}_{PLT} : \mathbf{L}^T \qquad \dot{P}_{ij} = \left(D_{PLT}\right)_{ijkl} L_{kl}$$

$$\dot{\mathbf{N}} = \overset{4}{\mathbf{D}}_{NL} : \mathbf{L} \qquad \dot{N}_{ij} = \left(D_{NL}\right)_{ijkl} L_{kl}$$

$$\dot{\mathbf{N}} = \overset{4}{\mathbf{D}}_{NLT} : \mathbf{L}^T \qquad \dot{N}_{ij} = \left(D_{NLT}\right)_{ijkl} L_{kl}$$

Energy conjugate
in Lagrangean configuration
but not in updated
Lagrangean configuration

- With some obvious relationships between the components of the 4th order stiffness tensors

$$\overset{4}{\mathbf{D}}_{NL} = \left(\overset{4}{\mathbf{D}}_{PL}\right)^{T12} \quad or \quad \left(D_{NL}\right)_{ijkl} = \left(D_{PL}\right)_{jikl}$$

$$\overset{4}{\mathbf{D}}_{NLT} = \left(\overset{4}{\mathbf{D}}_{PLT}\right)^{T12} \quad or \quad \left(D_{NLT}\right)_{ijkl} = \left(D_{PLT}\right)_{jikl}$$

$$\overset{4}{\mathbf{D}}_{PL} = \left(\overset{4}{\mathbf{D}}_{PLT}\right)^{T34} \quad or \quad \left(D_{PL}\right)_{ijkl} = \left(D_{PLT}\right)_{ijlk}$$

$$\overset{4}{\mathbf{D}}_{NL} = \left(\overset{4}{\mathbf{D}}_{NLT}\right)^{T34} \quad or \quad \left(D_{NL}\right)_{ijkl} = \left(D_{NLT}\right)_{ijlk}$$

- No wonder it gets confusing out there
- We have chosen to work primarily with the formulation in the 1PK and the velocity gradient

A change of notation

- Recall the relationship between the rate of the 1PK and velocity gradient and remember that both of these are neither tensors nor symmetric:

$$\dot{\mathbf{P}} = \overset{4}{\mathbf{D}}_{PL} : \mathbf{L}$$
$$\dot{P}_{ij} = \left(D_{PL}\right)_{ijkl} L_{kl}$$

$$\dot{\mathbf{P}} = \begin{pmatrix} \dot{P}_{11} & \dot{P}_{12} & \dot{P}_{13} \\ \dot{P}_{21} & \dot{P}_{22} & \dot{P}_{23} \\ \dot{P}_{31} & \dot{P}_{32} & \dot{P}_{33} \end{pmatrix} \qquad \mathbf{L} = \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \qquad \overset{4}{\mathbf{D}}_{PL} = \begin{pmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{D}_{13} \\ \mathbf{D}_{21} & \mathbf{D}_{22} & \mathbf{D}_{23} \\ \mathbf{D}_{31} & \mathbf{D}_{32} & \mathbf{D}_{33} \end{pmatrix}$$

- We can write this equivalently using 9-by-1 column matrices for P and L and a 9-by-9 matrix for the fourth order stiffness tensor:

$$\dot{\mathbf{P}}_{flat} = \overset{9-by-9}{\mathbf{D}_{PL}} \mathbf{L}_{flat}$$
$$\dot{P}_i = \left(D_{PL}\right)_{ij} L_j$$

$$\dot{\mathbf{P}}_{flat} = \left(\dot{P}_{11} \quad \dot{P}_{22} \quad \dot{P}_{33} \quad \dot{P}_{12} \quad \dot{P}_{13} \quad \dot{P}_{21} \quad \dot{P}_{23} \quad \dot{P}_{31} \quad \dot{P}_{32}\right)^T$$
$$\mathbf{L}_{flat} = \left(L_{11} \quad L_{22} \quad L_{33} \quad L_{12} \quad L_{13} \quad L_{21} \quad L_{23} \quad L_{31} \quad L_{32}\right)^T$$

- Note that this is NOT classical Voigt notation as no symmetry is exploited
- The double contractions now become simple matrix products

The General Bifurcation Criterion (GBC)

- The local formulation of Hill's criterion requires the second order work to be positive everywhere:

$$\Delta\mathbf{L} : \overset{4}{\mathbf{D}}_{PL} : \Delta\mathbf{L} > 0 \qquad \Delta L_{ij}\left(D_{PL}\right)_{ijkl}\Delta L_{kl} > 0$$

$$\Rightarrow$$

$$\mathbf{L} : \overset{4}{\mathbf{D}}_{PL} : \mathbf{L} > 0 \qquad L_{ij}\left(D_{PL}\right)_{ijkl} L_{kl} > 0 \qquad \mathbf{L}^T_{flat} \overset{9-by-9}{\mathbf{D}}_{PL} \mathbf{L}_{flat} > 0 \qquad L_i\left(D_{PL}\right)_{ij} L_j > 0$$

- For elastic/elasto-plastic materials the deviation from an equilibrium state can be replaced by an arbitrary velocity field as the stiffness matrix **D** has no direct dependency upon **L**
- The value of the quadratic form does not change if we replace the matrix of coefficients by its symmetric part :

$$\mathbf{L}^T_{flat} \overset{9-by-9}{\mathbf{D}}_{PL} \mathbf{L}_{flat} = \frac{1}{2}\mathbf{L}^T_{flat}\left[\overset{9-by-9}{\mathbf{D}}_{PL} + \left(\overset{9-by-9}{\mathbf{D}}_{PL}\right)^T\right]\mathbf{L}_{flat} > 0$$

- So we require that the symmetric part of the stiffness matrix **D** be positive definite
- A sufficient condition for a symmetric matrix to be positive definite is that all its eigenvalues be positive
- IFF **D** is positive definite in the initial (stress-free) state, then positive definiteness is lost when the first eigenvalue turns negative (= zero determinant)

$$\det\left(\overset{9-by-9}{\mathbf{D}}_{PL-SYM}\right) = \det\left[\overset{9-by-9}{\mathbf{D}}_{PL} + \left(\overset{9-by-9}{\mathbf{D}}_{PL}\right)^T\right] = 0$$

The Limit Point Bifurcation Criterion (LPBC)

- The LPBC follows from requiring that the same increment of the 1PK stress can co-exist with 2 velocity fields:

$$\Delta\dot{\mathbf{P}} = \overset{4}{\mathbf{D}}_{PL} : \Delta\mathbf{L} = \mathbf{0}$$
$$\Delta\dot{P}_{ij} = \left(D_{PL}\right)_{ijkl}\Delta L_{kl} = 0_{ij}$$

$$\dot{\mathbf{P}} = \overset{4}{\mathbf{D}}_{PL} : \mathbf{L} = \mathbf{0} \qquad \dot{\mathbf{P}}_{flat} = \overset{9-by-9}{\mathbf{D}_{PL}}\,\mathbf{L}_{flat} = \mathbf{0}_{flat}$$
$$\dot{P}_{ij} = \left(D_{PL}\right)_{ijkl} L_{kl} = 0_{ij} \qquad \dot{P}_i = \left(D_{PL}\right)_{ij} L_j = 0_i$$

- As with GBC this can be reformulated for an arbitrary velocity field if **D** has no direct dependency upon **L**
- We then obtain a homogeneous system of 9 equations with 9 unknowns, actually identical to the MFC
- There are 2 possibilities for the solutions of this homogeneous system:

$$\det\left(\overset{9-by-9}{\mathbf{D}_{PL}}\right) \neq 0 \Rightarrow \mathbf{L} = \mathbf{0} \;\; 1\left(trivial\right)\, solution \qquad \det\left(\overset{9-by-9}{\mathbf{D}_{PL}}\right) = 0 \Rightarrow \infty^{9-rank(\mathbf{D})} \; solutions \Rightarrow loss\ of\ uniqueness$$

- Note that GBC will always be fulfilled if LPBC is fulfilled since

$$\det\left(\overset{9-by-9}{\mathbf{D}_{PL-SYM}}\right) = \det\left(\overset{9-by-9}{\mathbf{D}_{PL}}\right)\det\left(I + \left(\overset{9-by-9}{\mathbf{D}_{PL}}\right)^{-1}\left(\overset{9-by-9}{\mathbf{D}_{PL}}\right)^{T}\right)$$

Remarks about GBC and LPBC

- Both criteria require to determine the zero-point of a matrix determinant:

$$GBC: \ \det\left(\overset{9-by-9}{\mathbf{D}_{PL-SYM}}\right) = \det\left[\overset{9-by-9}{\mathbf{D}_{PL}} + \left(\overset{9-by-9}{\mathbf{D}_{PL}}\right)^T\right] = 0 \qquad LPBC: \ \det\left(\overset{9-by-9}{\mathbf{D}_{PL}}\right) = 0$$

- The determinant of an n-by-n matrix has n! terms, so:

3-by3	6 terms
6-by-6	720 terms
9-by-9	362880 terms

- All 4 material formulations will result in the same instability criteria
- GBC defines a stable region in stress space, LPBC just defines a point where instability starts
- Note that the stable region defined by GBC is the region where all eigenvalues of **D** are positive, a positive determinant is a necessary but not sufficient condition
- We will endeavor to express both GBC and LPBC in function of the state of stress (principal stresses or stress invariants) and the material constants only, for this purpose it will be convenient to work in the principal system of the Cauchy stress

Objectivity of GBC and LPBC

- In general the transformation between a global fixed reference system (in which GBC and LPBC were derived) and the principal reference system of the Cauchy stress tensor is determined by a time dependent proper orthogonal matrix **Q**:

$$\mathbf{Q}^T(t) = \mathbf{Q}^{-1}(t) \qquad \dot{\mathbf{Q}} = \mathbf{\Gamma}\mathbf{Q} \qquad \mathbf{\Gamma}^T = -\mathbf{\Gamma} \qquad \det(\mathbf{Q}) = 1$$

- Although the rate of the 1PK stress as well as the velocity gradient are not objective quantities, the 4th order stiffness tensor transforms as an objective tensor:

$$\left(\overset{4}{\mathbf{D}}_{PL}\right)^{prinicpal} = (\mathbf{Q} \otimes \mathbf{Q})^{\overset{23}{T}} \overset{4}{\mathbf{D}}_{PL} \left(\mathbf{Q}^T \otimes \mathbf{Q}^T\right)^{\overset{23}{T}} \qquad (D_{PL})_{ijpo}^{principal} = Q_{ki} Q_{lj} (D_{PL})_{klmn} Q_{mp} Q_{no}$$

- And consequently (since the determinant of **Q** is unity):

$$\det\left(\overset{9-by-9}{\mathbf{D}}_{PL-SYM}\right) = \det\left(\overset{9-by-9}{\mathbf{D}}_{PL-SYM}\right)^{principal} \qquad \det\left(\overset{9-by-9}{\mathbf{D}}_{PL}\right) = \det\left(\overset{9-by-9}{\mathbf{D}}_{PL}\right)^{principal}$$

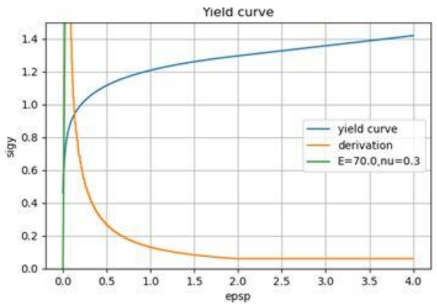
From GBC and LPBC to an ECRIT surface

- As we are allowed to transform into the principal system before computing the determinant, GBC and LPBC will result in an equation that can be solved for the critical value of the first principal stress for every state of stress:

$$\boldsymbol{\sigma}^{principal} = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \overset{\sigma_1 \neq 0}{=} \sigma_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix}$$

- Clearly LPBC and GBC now result in an equation for the critical value of the first principal stress:

$$\left.\begin{aligned} \det\left(\overset{9-by-9}{\mathbf{D}_{PL-SYM}}\right)^{principal} &= 0 \\ \det\left(\overset{9-by-9}{\mathbf{D}_{PL}}\right)^{principal} &= 0 \end{aligned}\right\} \Rightarrow \sigma_1^{ECRIT} = \begin{cases} f_{elastic}(a,b,E,\nu) \\ f_{plastic}(a,b,E,\nu,H(\varepsilon_p)) \end{cases}$$



- The hardening modulus H can be a constant or a monotonically decreasing function of the equivalent strain

From GBC and LPBC to an ECRIT surface

- IFF instability occurs in a plastic state, GBC and LPBC result in an instability surface giving a critical value of the equivalent plastic strain in function of triaxiality and Lode parameter
- The instability surface is obtained by eliminating the critical first principal stress between the instability condition and the hardening law:

$$\left.\begin{array}{r}\sigma_1^{ECRIT} = f\left(a,b,E,\nu,H\left(\varepsilon_p^{ECRIT}\right)\right)\\ \sigma_{vm}^2 = \left(\sigma_1^{ECRIT}\right)^2\left[1+a^2+b^2-a-b-ab\right] = \sigma_y^2\left(\varepsilon_p^{ECRIT}\right)\end{array}\right\}\Rightarrow \varepsilon_p^{ECRIT} = g(a,b)$$

- And replace the principal stress ratios by the classical invariants to comply with GISSMO:

$$\left.\begin{array}{r}\varepsilon_p^{ECRIT} = g(a,b)\\ -\dfrac{p}{\sigma_{vm}} = \dfrac{1}{3}\dfrac{1+a+b}{\sqrt{1+a^2+b^2-a-b-ab}}\dfrac{\sigma_1}{|\sigma_1|}\\ \dfrac{27s_1s_2s_3}{2\sigma_{vm}^3} = \dfrac{1}{2}\dfrac{(2-a-b)(2a-1-b)(2b-1-a)}{\left(\sqrt{1+a^2+b^2-a-b-ab}\right)^3}\end{array}\right\}\Rightarrow \boxed{\varepsilon_p^{ECRIT} = g\left(-\dfrac{p}{\sigma_{vm}},\dfrac{27s_1s_2s_3}{2\sigma_{vm}^3}\right)}$$

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Derivation of the 'PL' material law

- Compute the rate of the 1PK

$$\mathbf{P} = j\boldsymbol{\sigma}\mathbf{F}^{-T} \Rightarrow \dot{\mathbf{P}} = \dot{j}\boldsymbol{\sigma}\mathbf{F}^{-T} + j\boldsymbol{\sigma}\frac{d\mathbf{F}^{-T}}{dt} + j\dot{\boldsymbol{\sigma}}\mathbf{F}^{-T}$$

$$\dot{\mathbf{P}} = j\left[tr(\mathbf{D})\boldsymbol{\sigma}\mathbf{F}^{-T} - \boldsymbol{\sigma}\left(\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}\right)^{T} + \dot{\boldsymbol{\sigma}}\mathbf{F}^{-T}\right] \quad as \quad \frac{\dot{j}}{j} = tr(\mathbf{D})$$

$$\dot{\mathbf{P}} = j\left[tr(\mathbf{D})\boldsymbol{\sigma}\mathbf{F}^{-T} - \boldsymbol{\sigma}\left(\mathbf{F}^{-1}\mathbf{L}\mathbf{F}\mathbf{F}^{-1}\right)^{T} + \dot{\boldsymbol{\sigma}}\mathbf{F}^{-T}\right]$$

$$\dot{\mathbf{P}} = j\left[tr(\mathbf{D})\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{L}^{T} + \dot{\boldsymbol{\sigma}}\right]\mathbf{F}^{-T}$$

$$in\ updated\ Lagrangean:\ j = 1\ and\ \mathbf{F} = \mathbf{F}^{-T} = \mathbf{I}$$

$$\dot{\mathbf{P}} = \dot{\boldsymbol{\sigma}} + tr(\mathbf{D})\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{L}^{T}$$

… and bring in the Jaumann rate

$$\dot{\boldsymbol{\sigma}} = \overset{4}{\mathbf{C}}_{ep} : \mathbf{D} + \mathbf{W}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{W}$$

$$\dot{\mathbf{P}} = \overset{4}{\mathbf{C}}_{ep} : \mathbf{D} + \mathbf{W}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{W} + tr(\mathbf{D})\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{L}^{T}$$

$$\dot{\mathbf{P}} = \overset{4}{\mathbf{C}}_{ep} : \mathbf{D} + \mathbf{W}\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{W} + tr(\mathbf{D})\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{D} - \boldsymbol{\sigma}\mathbf{W}^{T}$$

$$\dot{\mathbf{P}} = \overset{4}{\mathbf{C}}_{ep} : \mathbf{D} + tr(\mathbf{D})\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{D} + \mathbf{W}\boldsymbol{\sigma}$$

- And use the definition to compute the fourth order stiffness tensor:

$$\dot{\mathbf{P}} = \overset{4}{\mathbf{D}}_{PL} : \mathbf{L} \Rightarrow \overset{4}{\mathbf{D}}_{PL} = \frac{\partial\dot{\mathbf{P}}}{\partial\mathbf{L}} = \overset{4}{\mathbf{C}}_{ep} + \frac{\partial}{\partial\mathbf{L}}\left(tr(\mathbf{D})\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{D} + \mathbf{W}\boldsymbol{\sigma}\right) = \overset{4}{\mathbf{C}}_{ep} + \overset{4}{\mathbf{T}}_{PL}$$

4^{th} order stiffness tensor as a 9-by-9 matrix

- Independently of the material law the second part of the 9-by-9 stiffness matrix (**T**) becomes:

$$\overset{9-by-9}{\mathbf{T}_{PL}} = \frac{\partial}{\partial \mathbf{L}}\left(tr(\mathbf{D})\boldsymbol{\sigma} - \boldsymbol{\sigma}\mathbf{D} + \mathbf{W}\boldsymbol{\sigma}\right) = \frac{1}{2}\begin{pmatrix} 0 & 2\sigma_{11} & 2\sigma_{11} & 0 & 0 & -2\sigma_{12} & 0 & -2\sigma_{13} & 0 \\ 2\sigma_{22} & 0 & 2\sigma_{22} & -2\sigma_{21} & 0 & 0 & 0 & 0 & -2\sigma_{23} \\ 2\sigma_{33} & 2\sigma_{33} & 0 & 0 & -2\sigma_{13} & 0 & -2\sigma_{32} & 0 & 0 \\ 2\sigma_{21} & 0 & 2\sigma_{21} & \sigma_{22}-\sigma_{11} & \sigma_{23} & -\sigma_{22}-\sigma_{11} & -\sigma_{13} & -\sigma_{23} & -\sigma_{13} \\ 2\sigma_{31} & 2\sigma_{31} & 0 & \sigma_{32} & -\sigma_{11}+\sigma_{33} & -\sigma_{32} & -\sigma_{12} & -\sigma_{33}-\sigma_{11} & -\sigma_{12} \\ 0 & 2\sigma_{12} & 2\sigma_{12} & -\sigma_{22}-\sigma_{11} & -\sigma_{23} & -\sigma_{22}+\sigma_{11} & \sigma_{13} & -\sigma_{23} & -\sigma_{13} \\ 2\sigma_{32} & 2\sigma_{32} & 0 & -\sigma_{31} & -\sigma_{21} & \sigma_{31} & -\sigma_{22}+\sigma_{33} & -\sigma_{21} & -\sigma_{33}-\sigma_{22} \\ 0 & 2\sigma_{13} & 2\sigma_{13} & \sigma_{32} & -\sigma_{33}-\sigma_{11} & -\sigma_{32} & -\sigma_{12} & \sigma_{11}-\sigma_{33} & \sigma_{12} \\ 2\sigma_{23} & 0 & 2\sigma_{23} & -\sigma_{31} & -\sigma_{21} & -\sigma_{31} & -\sigma_{22}-\sigma_{33} & \sigma_{21} & \sigma_{22}-\sigma_{33} \end{pmatrix}$$

Deriving this is index notation at a higher level

However, since **D** and **W** depend linearly on **L**, it is clear that only stress terms remain

- Transforming in the principal system of the Cauchy stress leads to a spectacular reduction:

$$\left(\overset{9-by-9}{\mathbf{T}_{PL}}\right)^{principal} = \frac{1}{2}\left(\begin{array}{ccc|cccccc} 0 & 2\sigma_{11} & 2\sigma_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\sigma_{22} & 0 & 2\sigma_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\sigma_{33} & 2\sigma_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \sigma_{22}-\sigma_{11} & 0 & -\sigma_{22}-\sigma_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sigma_{11}+\sigma_{33} & 0 & 0 & -\sigma_{33}-\sigma_{11} & 0 \\ 0 & 0 & 0 & -\sigma_{22}-\sigma_{11} & 0 & -\sigma_{22}+\sigma_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sigma_{22}+\sigma_{33} & 0 & -\sigma_{33}-\sigma_{22} \\ 0 & 0 & 0 & 0 & -\sigma_{33}-\sigma_{11} & 0 & 0 & \sigma_{11}-\sigma_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sigma_{22}-\sigma_{33} & 0 & \sigma_{22}-\sigma_{33} \end{array}\right)$$

The 9-by-9 matrix is split in an ‘upper’ 3-by-3 matrix and a ‘lower’ 6-by-6 matrix

No more coupling between incremental normal and shear terms

4^{th} order stiffness tensor as a 9-by-9 matrix

- In the case of isotropic hypo-elasticity the first (material) part (**C**) of the 9-by-9 stiffness matrix is independent of the reference system and can be written as:

$$\overset{9-by-9}{\mathbf{C}_{el}} = \begin{pmatrix} 2G+\lambda & \lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 2G+\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & \lambda & 2G+\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & G & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 & 0 & G & 0 \\ 0 & 0 & 0 & G & 0 & G & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G & 0 & G \\ 0 & 0 & 0 & 0 & G & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G & 0 & G \end{pmatrix} \quad \lambda = K - \frac{2G}{3}$$

Same uncoupling between normal and shear terms as in **T**

$$\overset{9-by-9}{\mathbf{C}_{ep}} = \overset{9-by-9}{\mathbf{C}_{el}} - \frac{2G}{\left(1+\frac{H}{3G}\right)} \frac{\mathbf{s}\otimes\mathbf{s}}{\|\mathbf{s}\|^2}$$

C has enough symmetry to reduce to a 6-by-6 matrix using Voigt notation
T however does not

- For the case of J2 plasticity we get the well known tangent stiffness matrix in the principal system:

$$\overset{9-by-9}{\mathbf{C}_{ep}} = \begin{pmatrix} 2G+\lambda & \lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 2G+\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & \lambda & 2G+\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & G & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 & 0 & G & 0 \\ 0 & 0 & 0 & G & 0 & G & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G & 0 & G \\ 0 & 0 & 0 & 0 & G & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & G & 0 & G \end{pmatrix} - \frac{3G}{\left(1+\frac{H}{3G}\right)\sigma_{vm}^2} \begin{pmatrix} s_{11}s_{11} & s_{11}s_{22} & s_{11}s_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ s_{22}s_{11} & s_{22}s_{22} & s_{22}s_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ s_{11}s_{33} & s_{22}s_{33} & s_{33}s_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} s_{11} = \dfrac{2\sigma_{11}-\sigma_{22}-\sigma_{33}}{3} \\ s_{22} = \dfrac{-\sigma_{11}+2\sigma_{22}-\sigma_{33}}{3} \\ s_{33} = \dfrac{-\sigma_{11}-\sigma_{22}+2\sigma_{33}}{3} \end{cases}$$

The lower 6-by-6 submatrix: LPBC / GBC cases

- After summation, the lower 6-by-6 submatrix is identical in the hypo-elastic and elasto-plastic cases:

$$\frac{1}{2}\begin{pmatrix} 2G+\sigma_{22}-\sigma_{11} & 0 & 2G-\sigma_{22}-\sigma_{11} & 0 & 0 & 0 \\ 0 & 2G+\sigma_{33}-\sigma_{11} & 0 & 0 & 2G-\sigma_{11}-\sigma_{33} & 0 \\ 2G-\sigma_{22}-\sigma_{11} & 0 & 2G-\sigma_{22}+\sigma_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2G+\sigma_{33}-\sigma_{22} & 0 & 2G-\sigma_{22}-\sigma_{33} \\ 0 & 2G-\sigma_{11}-\sigma_{33} & 0 & 0 & 2G-\sigma_{33}+\sigma_{11} & 0 \\ 0 & 0 & 0 & 2G-\sigma_{22}-\sigma_{33} & 0 & 2G-\sigma_{33}+\sigma_{22} \end{pmatrix}$$

- Due to its symmetry, this submatrix is also identical for LPBC and GBC
- Changing the order of the terms from (12-13-21-23-31-32) to (12-21-13-31-23-32) yields:

$$\frac{1}{2}\begin{pmatrix} 2G+\sigma_{22}-\sigma_{11} & 2G-\sigma_{22}-\sigma_{11} & 0 & 0 & 0 & 0 \\ 2G-\sigma_{22}-\sigma_{11} & 2G-\sigma_{22}+\sigma_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2G+\sigma_{33}-\sigma_{11} & 2G-\sigma_{11}-\sigma_{33} & 0 & 0 \\ 0 & 0 & 2G-\sigma_{11}-\sigma_{33} & 2G-\sigma_{33}+\sigma_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2G+\sigma_{33}-\sigma_{22} & 2G-\sigma_{22}-\sigma_{33} \\ 0 & 0 & 0 & 0 & 2G-\sigma_{22}-\sigma_{33} & 2G-\sigma_{33}+\sigma_{22} \end{pmatrix} \xrightarrow{\sigma=0} \begin{pmatrix} G & G & 0 & 0 & 0 & 0 \\ G & G & 0 & 0 & 0 & 0 \\ 0 & 0 & G & G & 0 & 0 \\ 0 & 0 & G & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & G \\ 0 & 0 & 0 & 0 & G & G \end{pmatrix}$$

- Allowing to write the determinant of this submatrix as a product of 3 determinants of 2-by-2 matrices
- Obviously this determinant is zero in the stress free state

LPBC criterion for J2 plasticity material law

- In the case of LPBC we identify non-uniqueness of the solution by setting the determinant of the '**PL**' stiffness matrix to zero:

$$
\det\left[\begin{pmatrix}
2G+\lambda & \lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
\lambda & 2G+\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
\lambda & \lambda & 2G+\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & G & G & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & G & G & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & G & G & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & G & G & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & G & G \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & G & G
\end{pmatrix}
-\frac{3G}{\left(1+\frac{H}{3G}\right)\sigma_{vm}^2}
\begin{pmatrix}
s_{11}s_{11} & s_{11}s_{22} & s_{11}s_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\
s_{22}s_{11} & s_{22}s_{22} & s_{22}s_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\
s_{11}s_{33} & s_{22}s_{33} & s_{33}s_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
+\frac{1}{2}
\begin{pmatrix}
0 & 2\sigma_{11} & 2\sigma_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\
2\sigma_{22} & 0 & 2\sigma_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\
2\sigma_{33} & 2\sigma_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sigma_{22}-\sigma_{11} & -\sigma_{22}-\sigma_{11} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\sigma_{22}-\sigma_{11} & -\sigma_{22}+\sigma_{11} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sigma_{33}-\sigma_{11} & -\sigma_{33}-\sigma_{11} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\sigma_{33}-\sigma_{11} & \sigma_{11}-\sigma_{33} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{33}-\sigma_{22} & -\sigma_{33}-\sigma_{22} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sigma_{33}-\sigma_{22} & \sigma_{22}-\sigma_{33}
\end{pmatrix}\right]=0
$$

$$
\det\begin{pmatrix}
\mathbf{A} & 0 & 0 & 0 \\
0 & \mathbf{D}_{12} & 0 & 0 \\
0 & 0 & \mathbf{D}_{31} & 0 \\
0 & 0 & 0 & \mathbf{D}_{23}
\end{pmatrix}=\det(\mathbf{A})\det(\mathbf{D}_{12})\det(\mathbf{D}_{31})\det(\mathbf{D}_{23})=0
$$

stress – free state :

$$
\boldsymbol{\sigma}=\mathbf{0}\Rightarrow\det(\mathbf{D}_{12})=\det(\mathbf{D}_{31})=\det(\mathbf{D}_{23})\equiv 0
$$

- Clearly this predicts loss of uniqueness in the stress-free state requiring better understanding

Loss of uniqueness need not imply localisation

- As the stiffness matrix is not positive definite in the stress-free state, the identification of the region where the solution is unique by determining the zero-points of the determinant is compromised for GBC
- Better understanding is gained from determining the non-unique solutions implied by LPBC around the stress free state, focusing on a single of the lower 2-by-2 submatrices we get:

$$\begin{pmatrix} \dot{P}_{12} \\ \dot{P}_{21} \end{pmatrix}^T = \begin{pmatrix} 2G & 2G \\ 2G & 2G \end{pmatrix} \begin{pmatrix} L_{12} \\ L_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow L_{12} = -L_{21}$$

- So any velocity field of the shape below would correspond to a maximum of the rate of **P**:

$$\det(\mathbf{D}_{12}) = 0 \Rightarrow \mathbf{L}_{flat} = (0 \quad 0 \quad 0 \quad L_{12} \quad L_{21} \quad 0 \quad 0 \quad 0 \quad 0) = L_{12}(0 \quad 0 \quad 0 \quad 1 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0)$$
$$\det(\mathbf{D}_{31}) = 0 \Rightarrow \mathbf{L}_{flat} = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad L_{13} \quad L_{31} \quad 0 \quad 0) = L_{31}(0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad -1 \quad 0 \quad 0)$$
$$\det(\mathbf{D}_{23}) = 0 \Rightarrow \mathbf{L}_{flat} = (0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad L_{23} \quad L_{32}) = L_{23}(0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad -1)$$

- And any linear combination of those solutions is also a solution to the linear homogeneous system as all 3 subdeterminants are zero in the stress-free state
- Clearly these are velocity fields corresponding to pure rigid body rotations and they do NOT imply any localisation, they would also be excluded by Hill's original GLOBAL uniqueness criterion

Critical principal stress values in plane stress

- Although the previous eliminates the concern about the stress-free state, other problems remain due to the local nature of both LPBC and GBC
- We will consider some examples next for plane states of stress:

$$\sigma_1 \neq 0 \qquad \sigma_3 = 0 \qquad \sigma_2 = k\sigma_1 \qquad k \leq 1$$

- Then each of the 3 determinants for the lower 2-by-2 submatrixes has 2 zero-points, for instance:

$$\det(\mathbf{D}_{12}) = 0 \Rightarrow (\sigma_1 \quad \sigma_2) = \begin{cases} (0 \quad 0) \\ \left(\dfrac{2G(1+k)}{1+k^2} \quad k\dfrac{2G(1+k)}{1+k^2} \right) \end{cases}$$

- Obviously the zero-point of the determinant of the upper 3-by-3 matrix may also yield the critical stress:

$$\det(\mathbf{A}) = 0$$

- And this evaluation is very different in the elastic and elasto-plastic cases

Loss of uniqueness due to spin components

- For a hypo-elastic material in uniaxial tension, the LPBC leads to:

$$\left.\begin{array}{r}\det(\mathbf{A})=0\rightarrow\sigma_1=\dfrac{E}{2\nu}\\ \det(\mathbf{D}_{12})=\det(\mathbf{D}_{31})=0\rightarrow\sigma_1=2G\end{array}\right\}\Rightarrow\sigma_1<\min\begin{pmatrix}\dfrac{E}{2\nu} & 2G\end{pmatrix}=\sigma_{critical}$$

- However, it is clear that a uniaxial state of stress in X-direction can co-exist with infinitely many strain fields that correspond to a rotation around X, for an elastic material:

$$\det(\mathbf{D}_{23})\equiv0\Rightarrow\begin{pmatrix}\sigma_1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\rightarrow\begin{pmatrix}L_{11} & 0 & 0\\ 0 & -\nu L_{11} & L_{23}\\ 0 & -L_{23} & -\nu L_{11}\end{pmatrix}$$

- The value of L23 can be freely chosen as long as L32=-L23, so we do lose uniqueness but obviously without localisation, the loss of uniqueness is due to the presence of a non-zero spin tensor
- A loss of uniqueness due to the spin tensor means that the same state of stress can co-exist with different velocity gradient fields, however it does NOT imply localisation

Loss of uniqueness around a state of pure shear

- We will now focus on the relevant lower 2-by-2 submatrix for a state of stress that deviates by an infinitesimal amount from pure shear in the 12-plane:

$$\sigma_{11} = \frac{2G(1+k)}{1+k^2} \quad \sigma_{22} = k\sigma_{11} = k\frac{2G(1+k)}{1+k^2} \qquad \textit{with} \quad k \to -1$$

$$\begin{pmatrix} \dot{P}_{12} \\ \dot{P}_{21} \end{pmatrix} = \begin{pmatrix} 2G-\sigma_{11}+\sigma_{22} & 2G-\sigma_{11}-\sigma_{22} \\ 2G-\sigma_{11}-\sigma_{22} & 2G+\sigma_{11}-\sigma_{22} \end{pmatrix} \begin{pmatrix} L_{12} \\ L_{21} \end{pmatrix} = \frac{4G}{1+k^2} \begin{pmatrix} k^2 & -k \\ -k & 1 \end{pmatrix} \begin{pmatrix} L_{12} \\ L_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow L_{21} = kL_{12} \to -L_{12}$$

$$\textit{note that} \ \ \mathbf{L} = L_{12}\begin{pmatrix} 1 \\ k \end{pmatrix} = \frac{L_{12}}{2}\begin{pmatrix} 1+k \\ 1+k \end{pmatrix} + \frac{L_{12}}{2}\begin{pmatrix} 1-k \\ k-1 \end{pmatrix} \overset{k\to-1}{\to} \frac{L_{12}}{2}\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \frac{L_{12}}{2}\begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

The non-unique solutions have a small pure shear component

- The non-unique solutions correspond to a combination of pure shear and rigid body rotation, as we approach pure shear they approach a pure rigid body rotation
- As we move away from pure shear, the non-uniqueness may imply localisation in a shear mode that occurs at very low levels of stress as (1+k) becomes a very small number

What exactly did Hill say ?

- Candidate velocity fields must not differ by just a rigid body rotation and must be compatible with the prescribed velocity boundary conditions:

5. Uniqueness Criterion

We regard the current distribution of stress in a body as given, together with the material properties at every point. For simplicity body forces are omitted since their mode of inclusion is sufficiently obvious. The nominal traction-rate $\dot{\mathbf{F}}$ is specified on a part S_F of the current surface and the velocity $\mathbf{v}$ on the remainder S_v. These conditions and the field equations (2), (3), (6) and (10) set a boundary-value problem for the internal velocity field.

Suppose that there could be two distinct solutions (not differing merely by a rigid-body motion when $S_v = 0$) and denote their difference by $\Delta\,\mathbf{v}$. Then, from

A sufficient condition for uniqueness is therefore that

$$\int \Delta \left\{ \frac{\partial E}{\partial\left(\partial v_j / \partial x_i\right)} \right\} \Delta \left(\frac{\partial v_j}{\partial x_i} \right) dV > 0 \qquad (20)$$

for all pairs of continuous velocity fields taking the prescribed values on S_v. By specializing (20) for pairs differing only infinitesimally, we see that it implies

$$\int c_{ijkl}\, \xi_{ij}\, \xi_{kl}\, dV > 0 \qquad (21)$$

- It was already clear from the singularity of the stiffness matrix in the stress-free state that formulating a local version of Hill's global criterion is not trivial
- Requiring the second order work to be positive in every point (GBC) shows that uniqueness can be lost and localisation can occur for very small stress values, the LPBC leads to similar conclusions
- However non unique solutions with a contribution of the spin components of the velocity gradient, although we feel they cannot be ignored, seem to be rare in practical applications, possibly due to the requirement of compatibility with the boundary conditions

GBC and LPBC for metals

- We can now evaluate GBC and LPBC by setting the determinants of the upper and lower 3-by-3 submatrices to zero and identifying the first zero point:

$$LPBC\ :\ \det\left[\begin{pmatrix} 2G+\lambda & \lambda & \lambda \\ \lambda & 2G+\lambda & \lambda \\ \lambda & \lambda & 2G+\lambda \end{pmatrix} - \frac{3G}{\left(1+\frac{H}{3G}\right)\sigma_{vm}^2}\begin{pmatrix} s_{11}s_{11} & s_{11}s_{22} & s_{11}s_{33} \\ s_{22}s_{11} & s_{22}s_{22} & s_{22}s_{33} \\ s_{33}s_{11} & s_{33}s_{22} & s_{33}s_{33} \end{pmatrix} + \begin{pmatrix} 0 & \sigma_{11} & \sigma_{11} \\ \sigma_{22} & 0 & \sigma_{22} \\ \sigma_{33} & \sigma_{33} & 0 \end{pmatrix}\right] = 0$$

$$GBC\ :\ \det\left[\begin{pmatrix} 2G+\lambda & \lambda & \lambda \\ \lambda & 2G+\lambda & \lambda \\ \lambda & \lambda & 2G+\lambda \end{pmatrix} - \frac{3G}{\left(1+\frac{H}{3G}\right)\sigma_{vm}^2}\begin{pmatrix} s_{11}s_{11} & s_{11}s_{22} & s_{11}s_{33} \\ s_{22}s_{11} & s_{22}s_{22} & s_{22}s_{33} \\ s_{33}s_{11} & s_{33}s_{22} & s_{33}s_{33} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & \sigma_{11}+\sigma_{22} & \sigma_{11}+\sigma_{33} \\ \sigma_{22}+\sigma_{11} & 0 & \sigma_{22}+\sigma_{33} \\ \sigma_{33}+\sigma_{11} & \sigma_{33}+\sigma_{22} & 0 \end{pmatrix}\right] = 0$$

Upper 3-by-3

GBC+LPBC:

$$\frac{1}{2}\begin{pmatrix} 2G+\sigma_{22}-\sigma_{11} & 2G-\sigma_{22}-\sigma_{11} & 0 & 0 & 0 & 0 \\ 2G-\sigma_{22}-\sigma_{11} & 2G-\sigma_{22}+\sigma_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2G+\sigma_{33}-\sigma_{11} & 2G-\sigma_{11}-\sigma_{33} & 0 & 0 \\ 0 & 0 & 2G-\sigma_{11}-\sigma_{33} & 2G-\sigma_{33}+\sigma_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2G+\sigma_{33}-\sigma_{22} & 2G-\sigma_{22}-\sigma_{33} \\ 0 & 0 & 0 & 0 & 2G-\sigma_{22}-\sigma_{33} & 2G-\sigma_{33}+\sigma_{22} \end{pmatrix} \xrightarrow{\sigma=0} \begin{pmatrix} G & G & 0 & 0 & 0 & 0 \\ G & G & 0 & 0 & 0 & 0 \\ 0 & 0 & G & G & 0 & 0 \\ 0 & 0 & G & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & G \\ 0 & 0 & 0 & 0 & G & G \end{pmatrix}$$

Lower 6-by-6

- The evaluation is an almost trivial task

Solutions for plane stress

(LPBC, Elastoplasticity, E≈H)

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} = 0 \end{pmatrix}$$

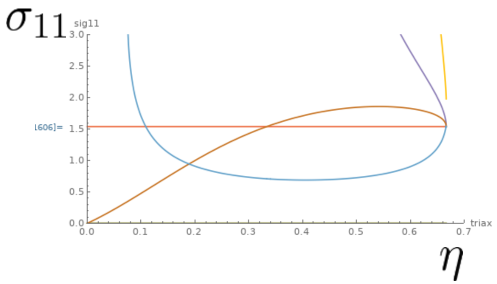
$$\dot{\mathbf{P}}_{flat} = \left(\dot{P}_{11},\, \dot{P}_{22},\, \dot{P}_{33},\, \dot{P}_{12},\, \dot{P}_{21},\, \dot{P}_{13},\, \dot{P}_{31},\, \dot{P}_{23},\, \dot{P}_{32}\right)$$

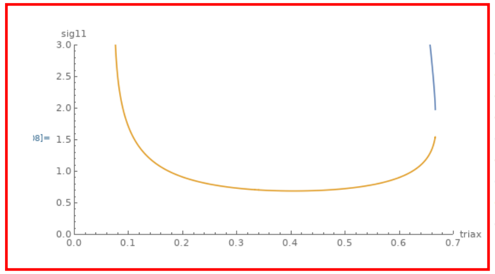
$$\mathbf{L}_{\mathbf{flat}} = \left(\varepsilon_{11},\, \varepsilon_{22},\, \varepsilon_{33},\, \varepsilon_{12},\, \varepsilon_{21},\, \varepsilon_{13},\, \varepsilon_{31},\, \varepsilon_{23},\, \varepsilon_{32}\right)^T$$

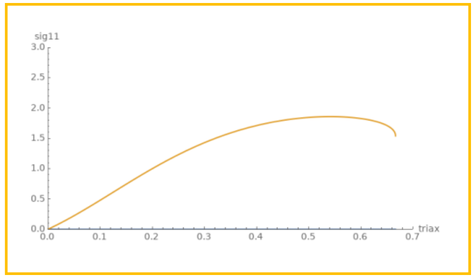
$$\mathbf{C}^{PL}_{9x9} = \frac{\partial \dot{\mathbf{P}}_{flat}}{\partial \mathbf{L}_{flat}}$$

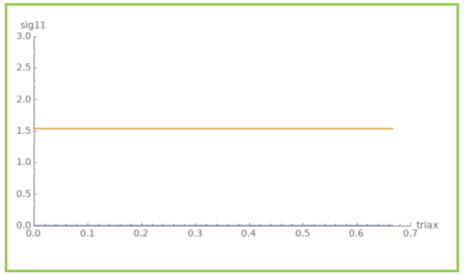
$$\left(\begin{array}{ccccccccc}
2G + lamda - \frac{G^2 (-2\,sig11+sig22)^2}{(3G+H)(sig11^2 - sig11\,sig22+sig22^2)} & lamda + sig11 + \frac{G^2 (sig11-2\,sig22)(2\,sig11-sig22)}{(3G+H)(sig11^2 - sig11\,sig22+sig22^2)} & lamda + sig11 + \frac{G^2 (2\,sig11-sig22)(sig11+sig22)}{(3G+H)(sig11^2 - sig11\,sig22+sig22^2)} & 0 & 0 & 0 & 0 & 0 & 0 \\
lamda + sig22 + \frac{G^2 (sig11-2\,sig22)(2\,sig11-sig22)}{(3G+H)(sig11^2 - sig11\,sig22+sig22^2)} & 2G + lamda - \frac{G^2 (sig11-2\,sig22)^2}{(3G+H)(sig11^2 - sig11\,sig22+sig22^2)} & lamda + sig22 - \frac{G^2 (sig11-2\,sig22)(sig11+sig22)}{(3G+H)(sig11^2 - sig11\,sig22+sig22^2)} & 0 & 0 & 0 & 0 & 0 & 0 \\
lamda + \frac{G^2 (2\,sig11-sig22)(sig11+sig22)}{(3G+H)(sig11^2 - sig11\,sig22+sig22^2)} & lamda - \frac{G^2 (sig11-2\,sig22)(sig11+sig22)}{(3G+H)(sig11^2 - sig11\,sig22+sig22^2)} & 2G + lamda - \frac{G^2 (sig11+sig22)^2}{(3G+H)(sig11^2 - sig11\,sig22+sig22^2)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & G + \frac{1}{2}(-sig11+sig22) & G - \frac{sig11}{2} - \frac{sig22}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & G - \frac{sig11}{2} - \frac{sig22}{2} & G + \frac{sig11-sig22}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & G - \frac{sig11}{2} & G - \frac{sig11}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & G - \frac{sig11}{2} & G + \frac{sig11}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & G - \frac{sig22}{2} & G - \frac{sig22}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & G - \frac{sig22}{2} & G + \frac{sig22}{2}
\end{array}\right)$$

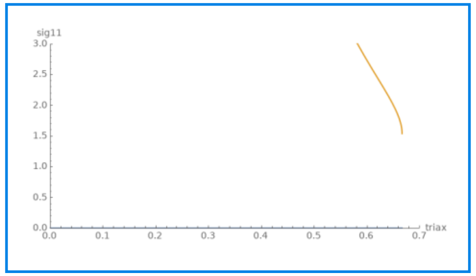
$$det(\mathbf{C}^{PL}_{9x9}) = det(\mathbf{C}^{PL}_{3x3,ii}) * det(\mathbf{C}^{PL}_{2x2,12}) * det(\mathbf{C}^{PL}_{2x2,13}) * det(\mathbf{C}^{PL}_{2x2,23}) = 0$$









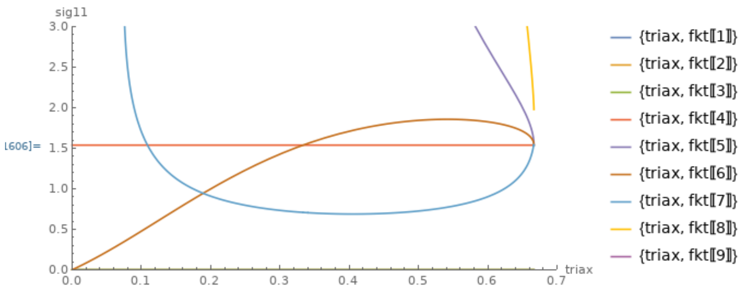


Analytical solutions for the plane stress state

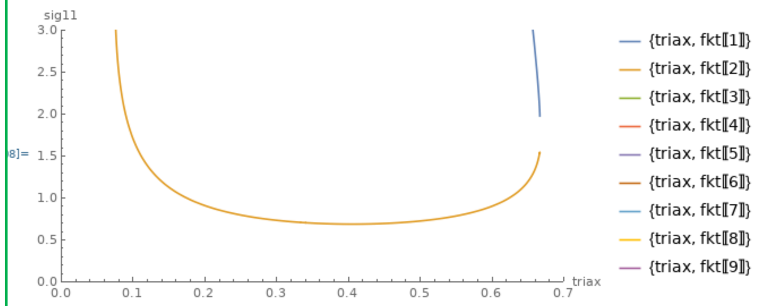
LPBC, Elastoplasticity, E=2, H=0.7, nu=0.3

$$det(\mathbf{C}^{PL}_{9x9}) \;=\; det(\mathbf{C}^{PL}_{3x3up}) \;*\; det(\mathbf{C}^{PL}_{6x6down}) = 0$$

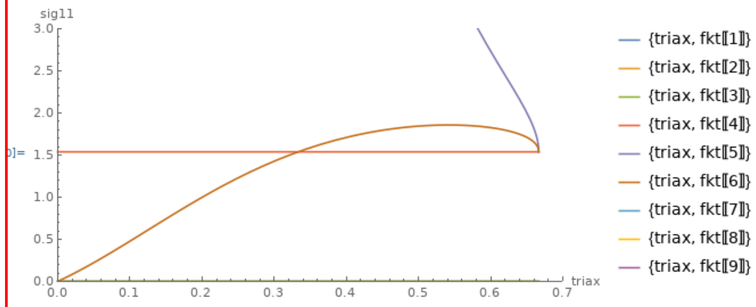
$$det(\mathbf{C}^{PL}_{9x9}) = 0$$



$$det(\mathbf{C}^{PL}_{3x3up}) = 0$$



$$det(\mathbf{C}^{PL}_{6x6down}) = 0$$

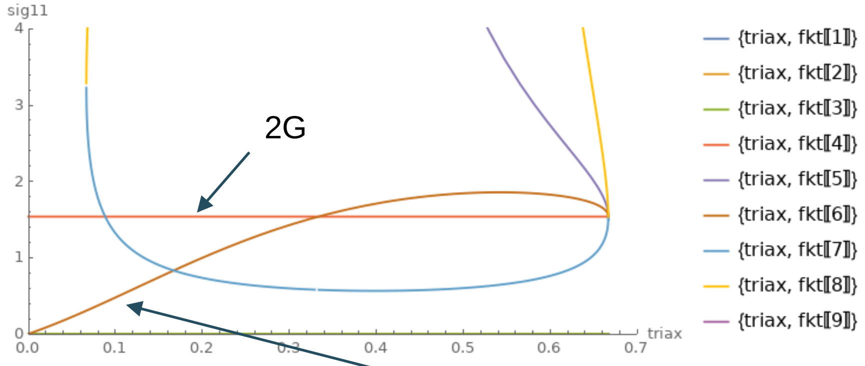


However non unique solutions with a contribution of the spin components of the velocity gradient, although we feel they cannot be ignored, seem to be rare in practical applications, possibly due to the requirement of compatibility with the boundary conditions

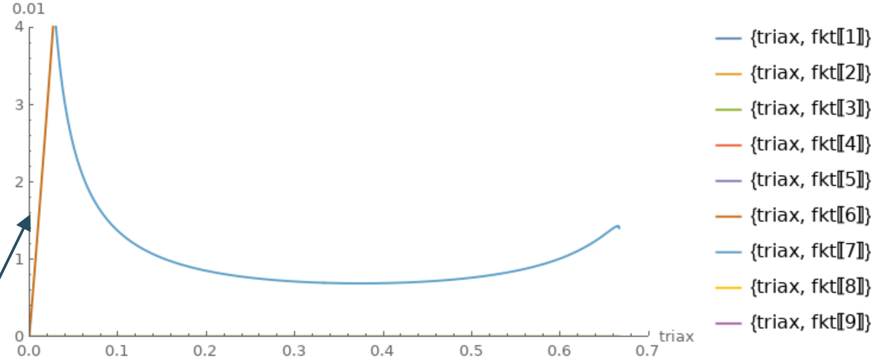
Analytical solutions for the plane stress state

LPBC, Elastoplasticity, E=2 and E=70, H=0.7, nu=0.3

LPBC (E=2 GPa)



LPBC (E=70 GPa)



Here the root from the lower 6-by-6 matrix is decisive!

- In addition to the argumentation of the previous slide we can see that for materials with a Young's modulus that far exceeds the hardening modulus, the region where the lower 6x6 matrix is decisive becomes very small.
- For both reasons we therefore feel justified in relaxing both GBC and LPBC by not considering the lower6x6-solutions for realistic metallic materials (E>>H)

GBC and LPBC, elastic and triaxiality=0.333…

- As an example compute all the roots for LPBC and GBC under uniaxial loads in the hypo-elastic case:

$$LPBC\ :upper$$

$$\det\left[\begin{pmatrix} 2G+\lambda & \lambda & \lambda \\ \lambda & 2G+\lambda & \lambda \\ \lambda & \lambda & 2G+\lambda \end{pmatrix} + \begin{pmatrix} 0 & \sigma_{11} & \sigma_{11} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = 0$$

$$3G + \frac{2G^2}{\lambda} = \sigma_{11} \rightarrow \sigma_{11} = \frac{9KG}{3K-2G} = \frac{E}{2\nu}$$

$$GBC\ :upper$$

$$\det\left[\begin{pmatrix} 2G+\lambda & \lambda & \lambda \\ \lambda & 2G+\lambda & \lambda \\ \lambda & \lambda & 2G+\lambda \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & \sigma_{11} & \sigma_{11} \\ \sigma_{11} & 0 & 0 \\ \sigma_{11} & 0 & 0 \end{pmatrix}\right] = 0$$

$$\sigma_{11}^2 + 4\lambda\sigma_{11} - 8G^2 - 12G\lambda = 0 \rightarrow \sigma_{11} = \begin{cases} -2\lambda + 2\sqrt{\lambda^2 + 2G^2 + 3G\lambda} \\ -2\lambda - 2\sqrt{\lambda^2 + 2G^2 + 3G\lambda} \end{cases}$$

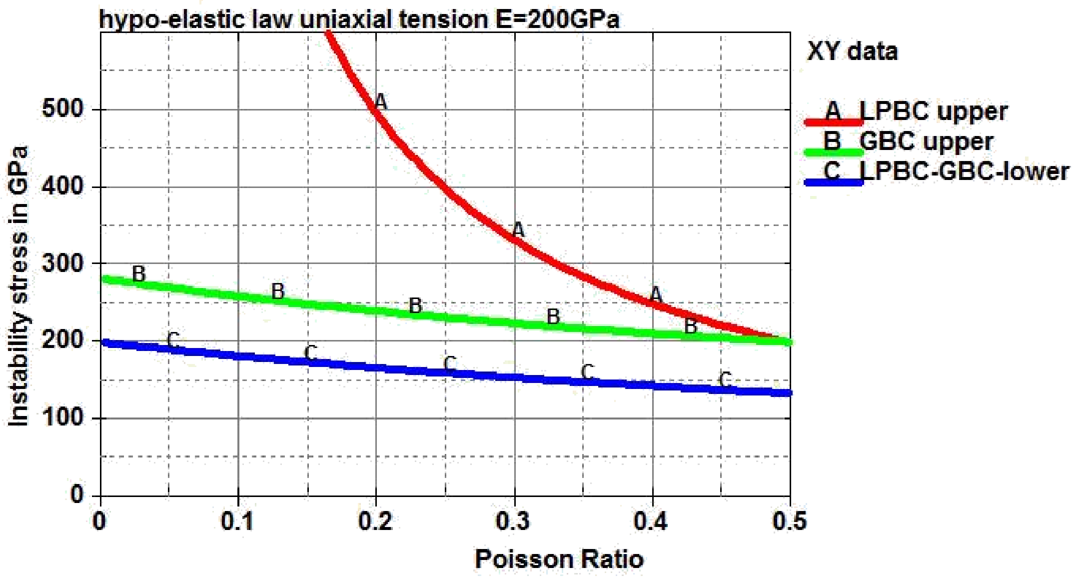
$$LPBC\ \ and\ \ GBC\ :\ lower$$

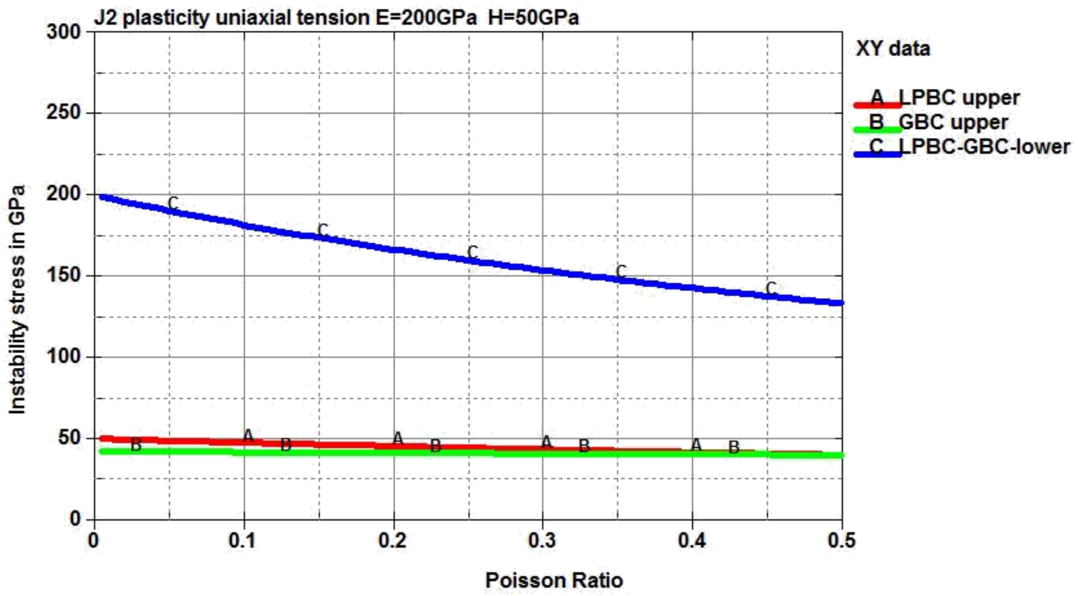
$$\frac{1}{2}\begin{pmatrix} 2G-\sigma_{11} & 2G-\sigma_{11} & 0 & 0 & 0 & 0 \\ 2G-\sigma_{11} & 2G+\sigma_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 2G-\sigma_{11} & 2G-\sigma_{11} & 0 & 0 \\ 0 & 0 & 2G-\sigma_{11} & 2G+\sigma_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2G & 2G \\ 0 & 0 & 0 & 0 & 2G & 2G \end{pmatrix} \rightarrow \frac{1}{2}(2G-\sigma_{11})(2G+\sigma_{11}-2G+\sigma_{11}) = 0 \rightarrow \begin{cases} \sigma_{11} = 0 \\ \sigma_{11} = 2G \end{cases}$$

Zero root is discarded as the corresponding velocity field is a pure rigid body rotation

- Note that the lower determinant yields the lowest prediction for the critical principal stress in the elastic case

Analytical solutions for the uniaxial tension case





$$LPBC: \ \sigma_1 = \frac{E}{2\nu}$$

$$GBC: \ \sigma_1 = \frac{2E}{\sqrt{2}\sqrt{1-\nu}+2\nu}$$

$$lower: \sigma_1 = \frac{E}{1+\nu} = 2G$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$LPBC: \ \sigma_1 = \frac{EH}{E+2\nu H}$$

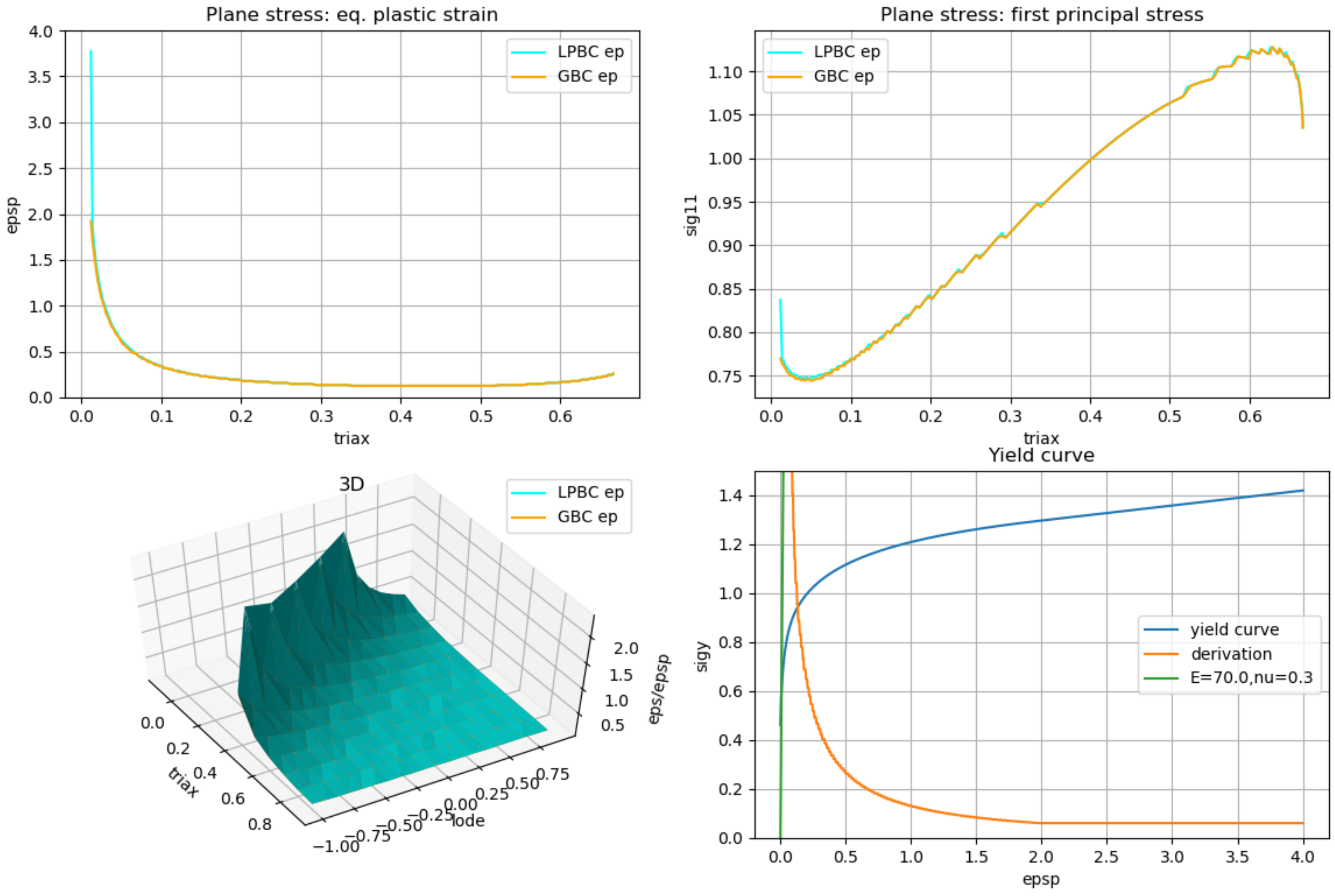
$$GBC: \ \sigma_1 = \frac{2EH}{E+2H\nu+\sqrt{(E+H)(E+2H-2H\nu)}}$$

$$lower: \ \sigma_1 = \frac{E}{1+\nu} = 2G$$

Comparison of GBC and LPBC

J2 elastoplasticity (real material DP800)

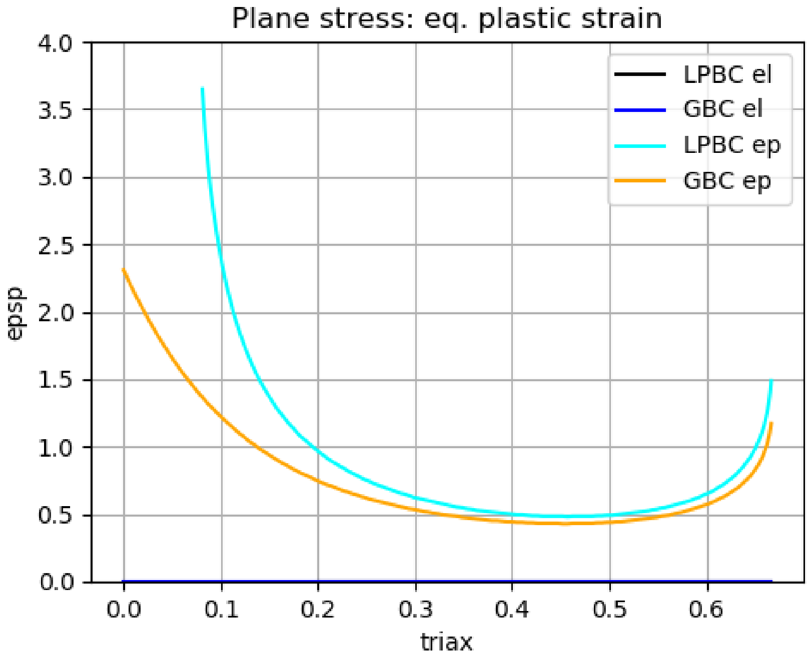
- Elastic properties
  E = 210 GPa
  nu = 0.3
- Plastic properties
  yield curve
- LPBC and GBC quite similar!

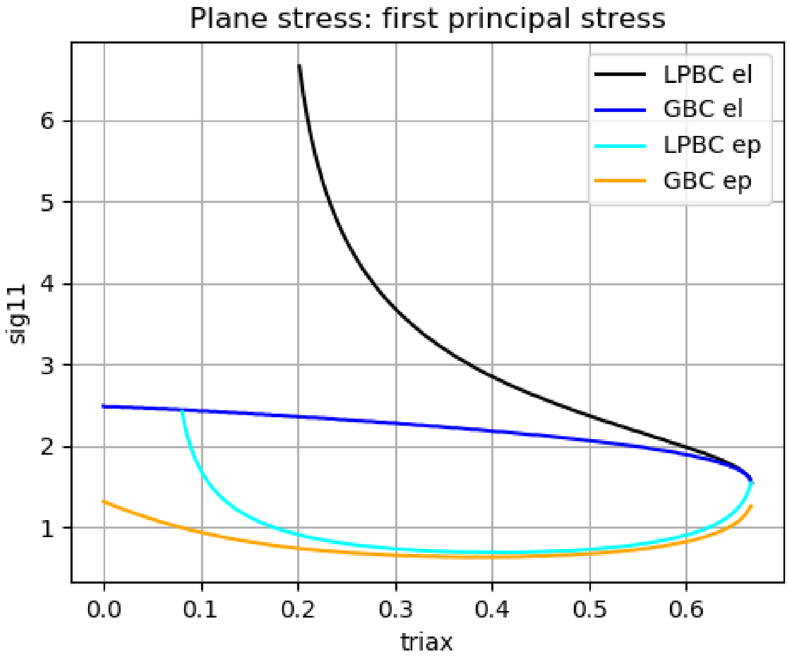


Comparison of GBC and LPBC

J2 elastoplasticity (artificial material properties, soft material)

- Elastic properties
  E = 2 GPa
  nu = 0.3
- Plastic properties
  bilinear yield curve
  sigy = 0.2 GPa
  H = 0.9 GPa





- LPBC and GBC quite different!

Analytical solution for unixial tension

Hypoelasticity:

$$LPBC : \sigma_1 = \frac{E}{2\nu}$$

$$GBC : \sigma_1 = \frac{2E}{\sqrt{2}\sqrt{1-\nu} + 2\nu}$$

Elastoplasticity:

$$LPBC : \sigma_1 = \frac{EH}{E + 2\nu H}$$

$$GBC : \sigma_1 = \frac{2EH}{E + 2H\nu + \sqrt{(E+H)(E+2H-2H\nu)}}$$

Some remarks

- Hill‘s criterion identifies the loss of uniqueness of the solution to the equations of elasto-plasticity
- It is undisputable but global and cumbersome to check
- GBC is a local version of Hill‘s criterion and amounts to an overconservative estimate
- Moreover, loss of uniqueness is a clear mathematical notion but does not always imply strain localisation
- LPBC is a local criterion for loss of uniqueness for the material law without considering boundary conditions, for elastic and elasto-plastic material laws the LPBC coincides with the MFC (maximum force criterion) and familiar criteria such as Considere and Swift are subsets of LPBC
- For example, we have shown that if we ignore the lower submatrices the LPBC is identical to a 3D generalisation of Swift 1952 if one correctly considers the elastic strains

Conclusions

- We have derived 2 analytical criteria for the onset of strain localisation
- But as we say in Detroit: Will it run ???

- Before we put confidence in the predicted instability surfaces by our modified GBC/LPBC we need to get confirmation from comparisons with experimental or numerical results
- In other words: Are LPBC or GBC consistent with J2-plasticity? Or even more restrictive: Is one of them consistent with MAT_024?
- Will models based on MAT_024 show a start of localisation of plastic deformation under the conditions that our equations predict?
- This is the object of our second presentation today

If it runs.....

- ECRIT has been used as a fitting parameter in GISSMO, as the real physical criterion was not known, the curve/surface were often adapted in an arbitrary way in order to fit specific test data
- The availability of a physical criterion will take this degree of freedom away from users but definitely bring the simulation results closer to the physics
- In other words some local fits may be worse but the overall reliability of the failure predictions produced by the model will improve
- Bottom line: Preparation of 2D and 3D GISSMO data will become easier as ECRIT will be derived automatically from the yield curve of the material

Thank You

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