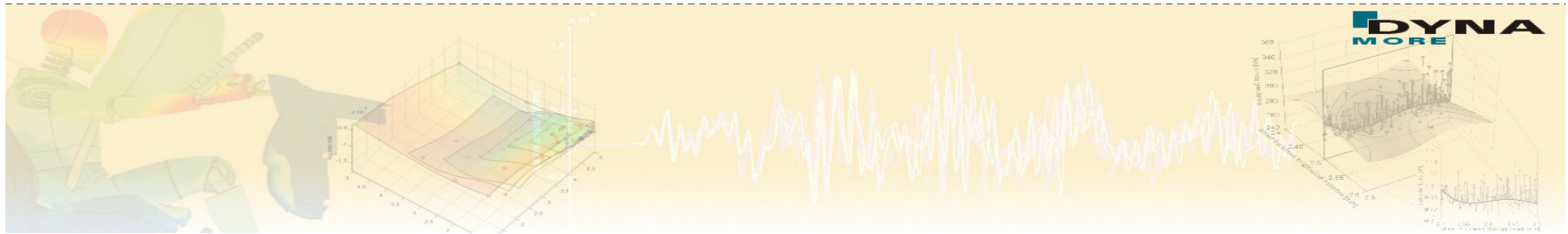


Neue Entwicklungen in LS-OPT 4.0 – Ausblick auf Version 4.1

New Developments in LS-OPT 4.0 – Outlook V4.1

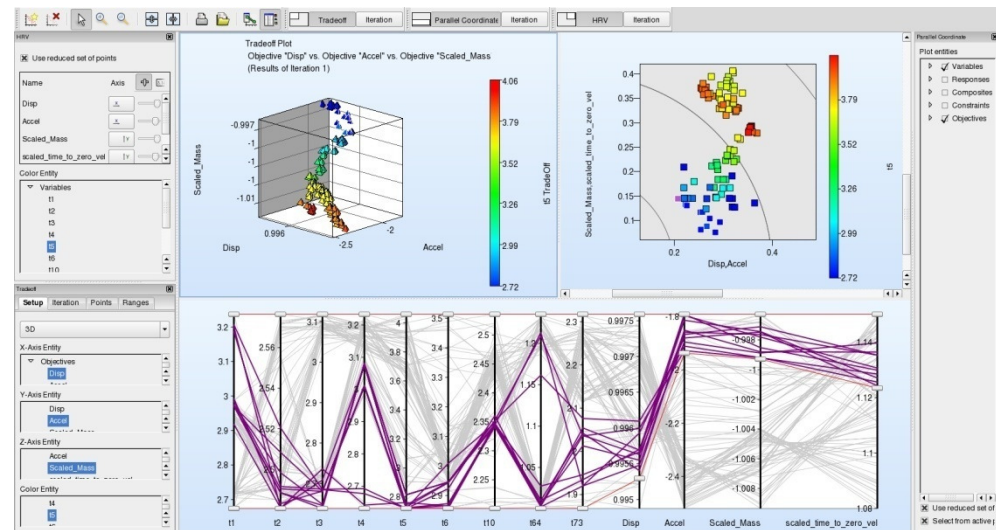
Heiner Müllerschön
hm@dynamore.de

DYNAmore GmbH
Industriestraße 2
70565 Stuttgart
<http://www.dynamore.de>



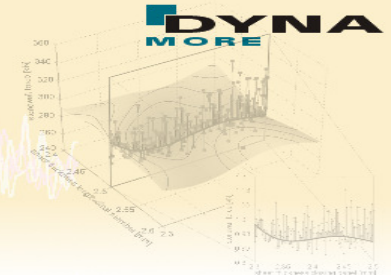
Overview

- Introduction/Features of current Version 4.0
- Methodologies – Optimization
- Methodologies - Robustness
- Examples - Optimization
- Examples – Robustness
- Outlook Version 4.1



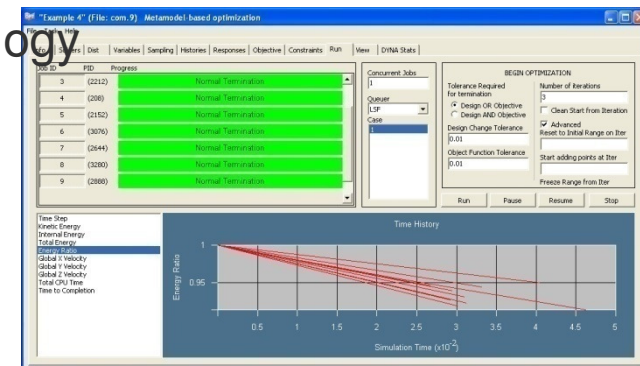
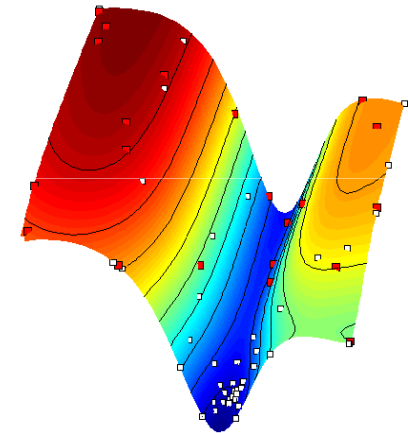
Introduction / Features

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About LS-OPT

- LS-OPT is a product of LSTC, current version: V4.0
 - *major new release with many new feature particularly within the viewer (visualization of results)*
 - next presentation “Methoden zur Visualisierung von Ergebnissen aus Optimierungs- und DOE-Studien“
- Beta version 4.1 available by end of this year
- LS-OPT can be linked to any simulation code – stand alone optimization software
- Perfect suitable in combination with LS-DYNA
- Two main products LS-OPT and LS-OPT/Topology



Introduction / Features

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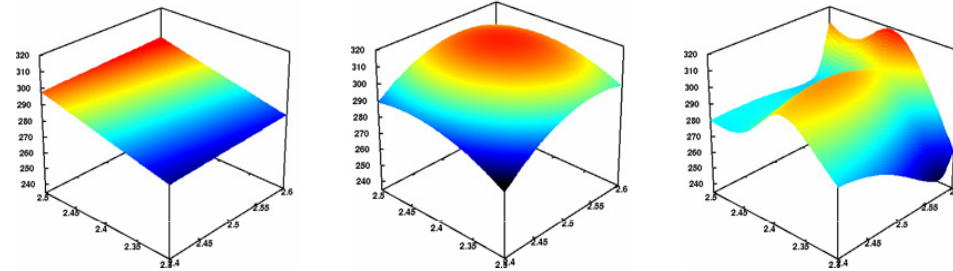


LS-OPT – Overview Methodologies

■ Successive Response Surface Method (SRSM)

■ Meta-Models

- *Polynomials*
- *Radial Basis Functions*
- *Neural Nets (FFNN)*



■ Genetic Algorithm (MOGA->NSGA-II)

■ Multidisciplinary optimization (MDO)

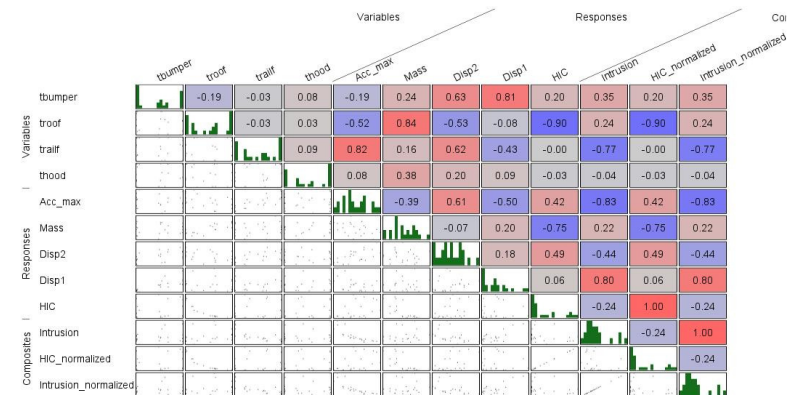
■ Multi-Objective Optimization (Pareto Front)

■ DOE-Studies (ANOVA, Sobol)

■ Parameter/System Identification

■ Stochastic/Probabilistic Analysis

■ Monte Carlo Analysis using Meta-Models



Introduction / Features

- Introduction/Features
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DYNA
MORE

LS-OPT – Overview Methodologies

■ Mixed Discrete-Continuous Optimization

- *Specify sets of discrete variables (e.g. sheet thicknesses)*

■ Robust Parameter Design (RDO)

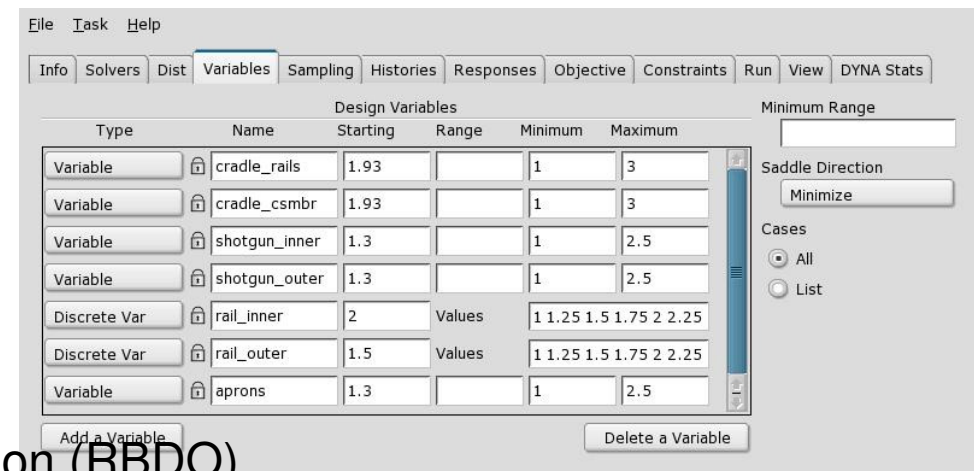
- *Improve/Maximizing the robustness of the optimum*

■ Reliability Based Design Optimization (RBDO)

- *Improve failure probability of optimum*

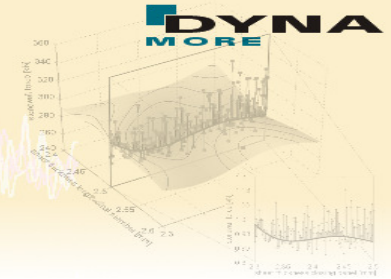
■ Visualization of Stochastic Results

- *Confidence Intervals, reliability quantities*
- *Fringe of statistic results on the FE-Model*



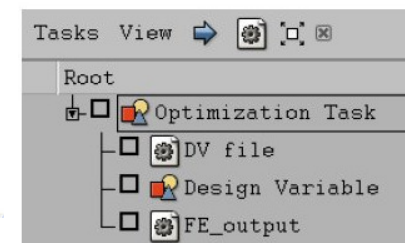
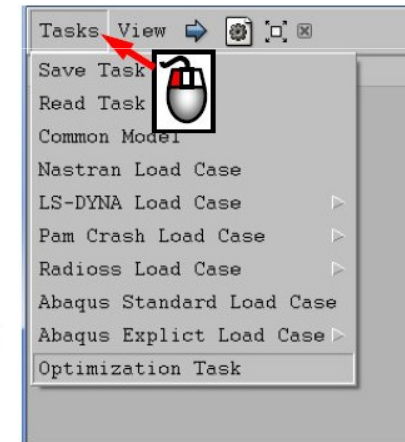
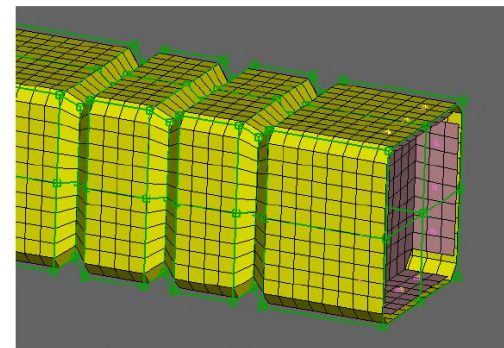
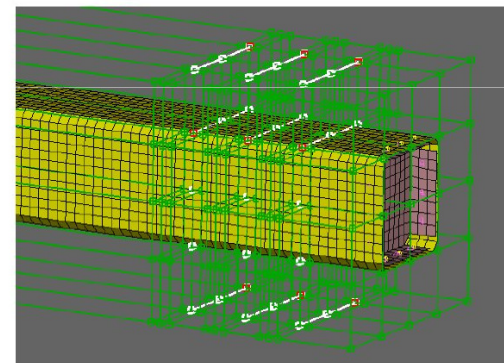
Introduction / Features

- Introduction/Features
- Methods – Optimization
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- Examples - Optimization
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- Version 4.1/ Outlook



About LS-OPT

- Job Distribution - Interface to Queuing Systems
 - *PBS, LSF, LoadLeveler, SLURM, AQS, etc.*
 - *Retry of failed queuing (abnormal termination)*
- LS-OPT might be used as a “Process Manager”
- Shape Optimization
 - *Interface to ANSA, HyperMorph, DEP-Morpher, SFE-Concept*
- META Post interface
 - *Allows extraction of results from any package (Abaqus, NASTRAN, ...) supported by META Post (ANSA package)*



Introduction / Features

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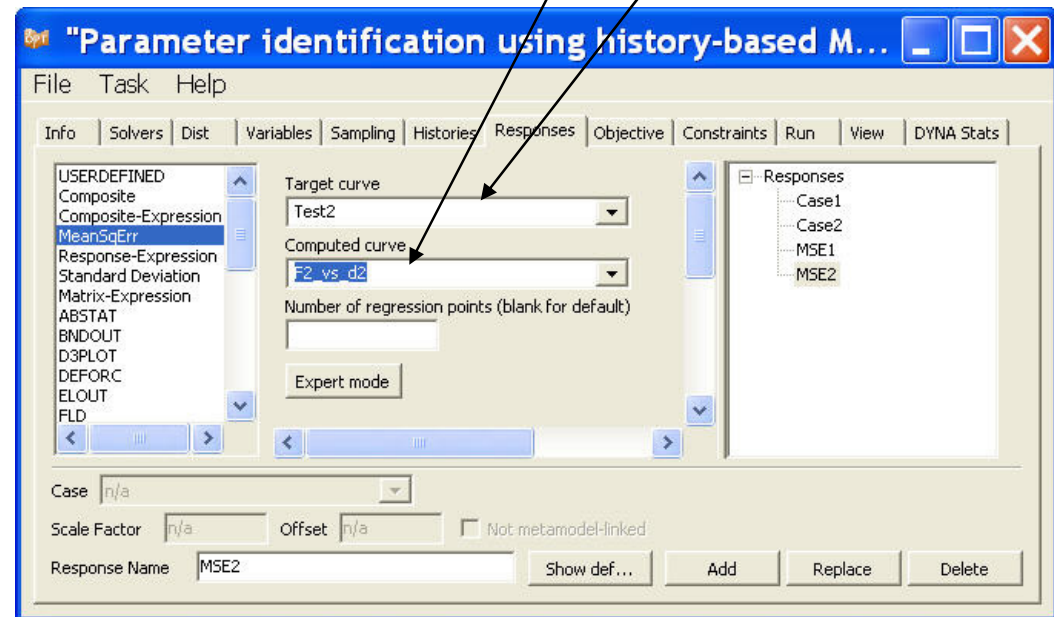


About LS-OPT

■ Parameter Identification Module

- *Handles "continuous" test curves*
- *Automated use of test results to calibrate materials/systems*
- *Simplify input for system identification applications*
- *Visualization of test and simulation curve to compare*
- *Confidence intervals for individual parameters in parameter identification*

$$\frac{1}{P} \sum_{p=1}^P W_i \left(\frac{F_i(\mathbf{x}) - G_i}{s_i} \right)^2$$



Introduction / Features

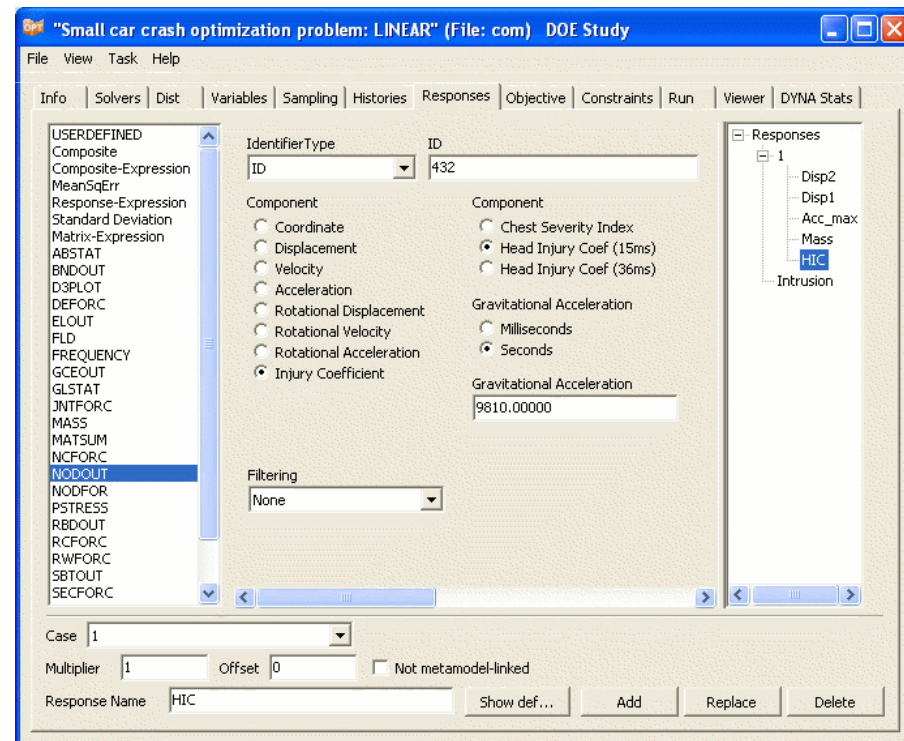
- Introduction/Features
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About LS-OPT

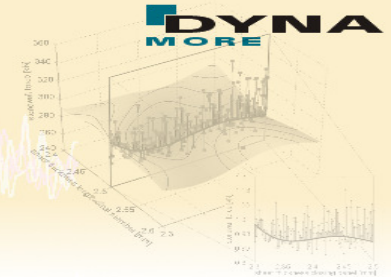
■ LS-DYNA Integration

- *Checking of Dyna keyword files (*DATABASE_)*
- *Importation of design parameters from Dyna keyword files (*PARAMETER_)*
- *Monitoring of LS-DYNA progress*
- *Result extraction of most LS-DYNA response types*
- *D3plot compression (node and part selection)*



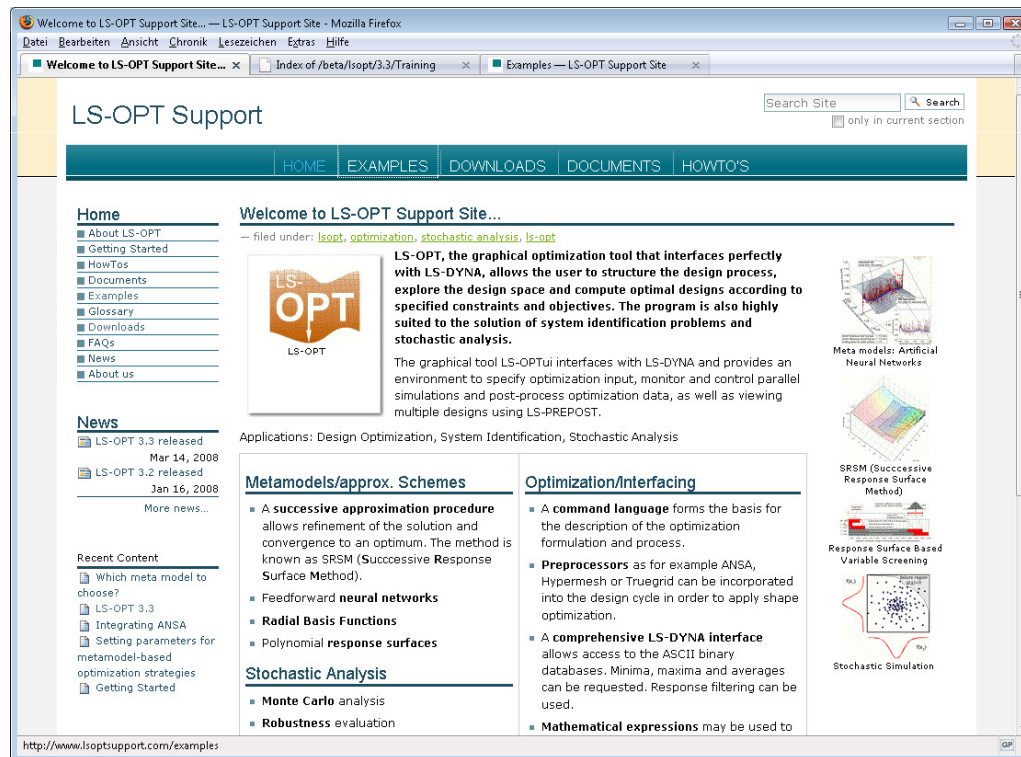
Introduction / Features

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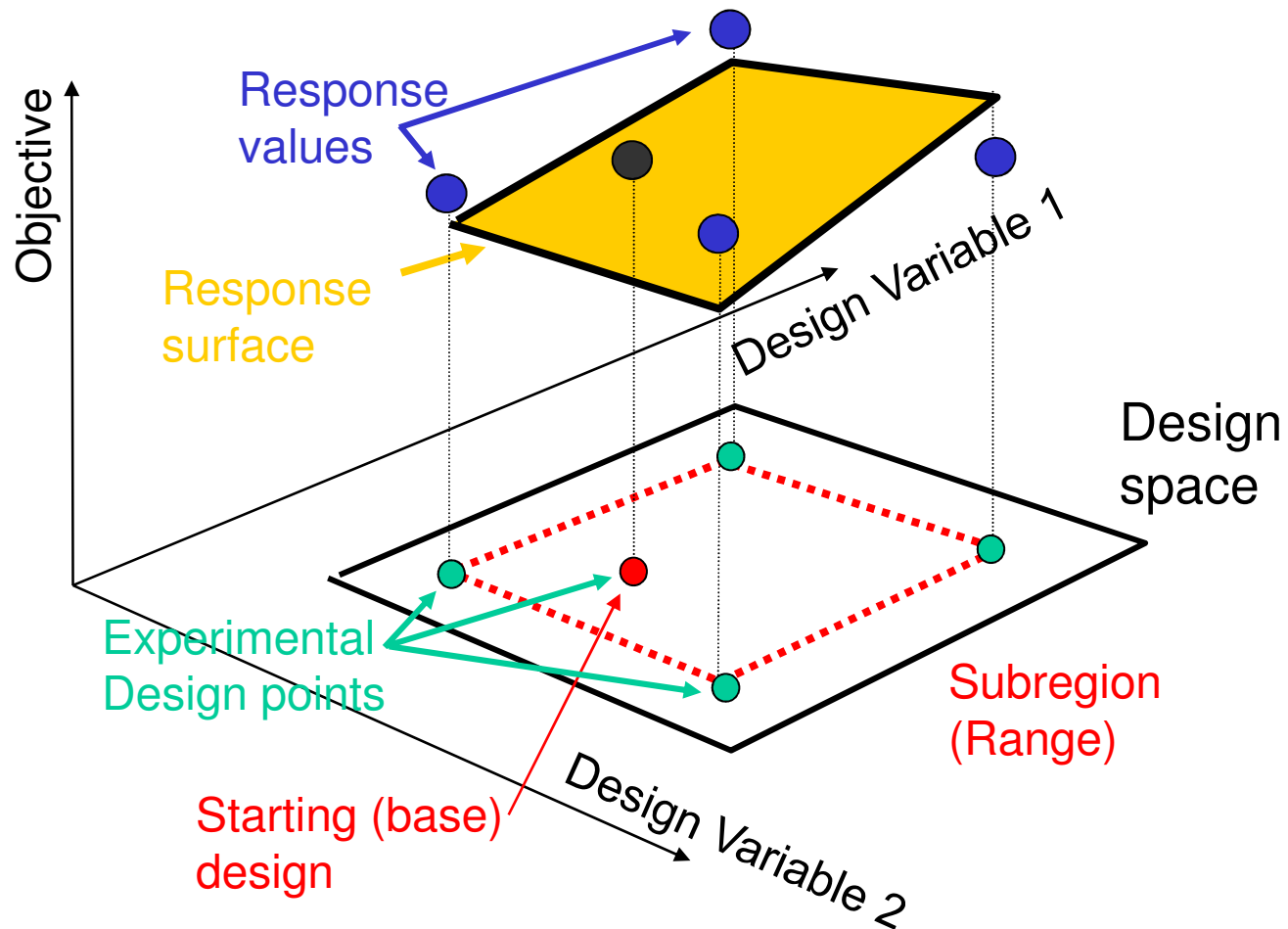


About LS-OPT

- LS-OPT Supportwebpage -> www.lsoptsupport.com
- *Many examples, tutorials, FAQs, HowTos...*



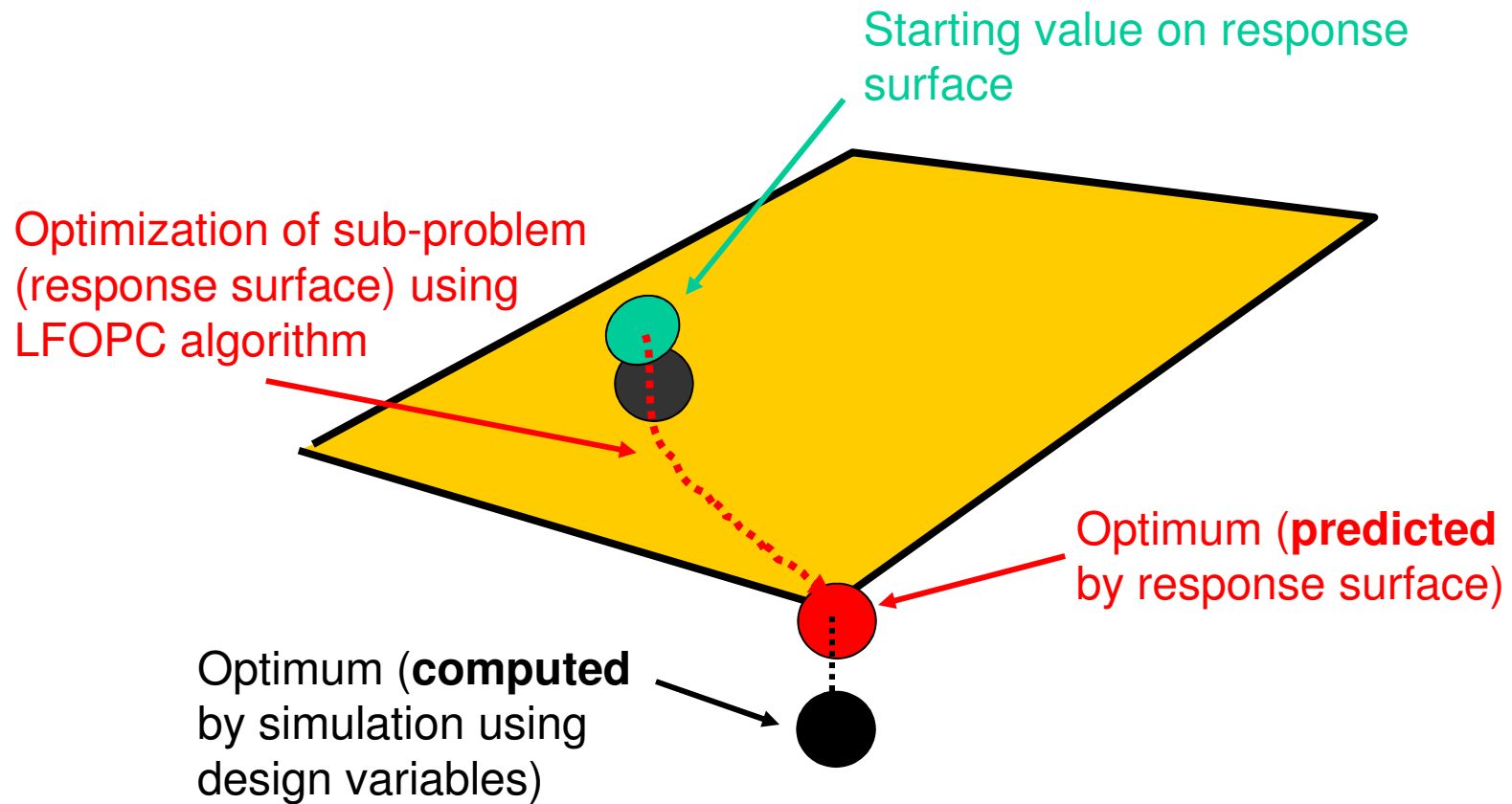
Response Surface Methodology - Optimization Process



Methods - Optimization

- Introduction/Features
- **Methods – Optimization**
- Methods - Robustness
- Examples - Optimization
- Examples - Robustness
- Version 4.1 / Outlook

Find an Optimum on the Response Surface (one iteration)

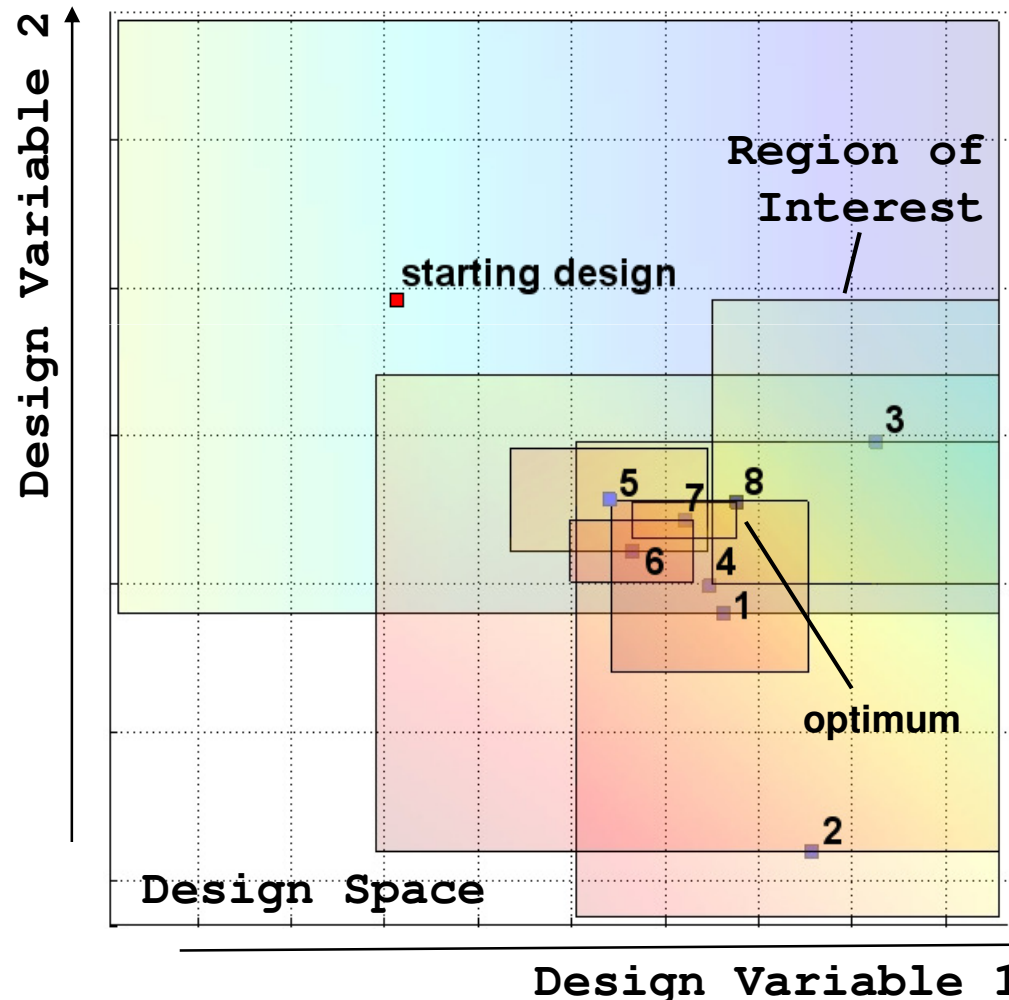


Methods - Optimization

- Introduction/Features
- Methods – Optimization**
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- Examples - Optimization
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- Version 4.1 / Outlook

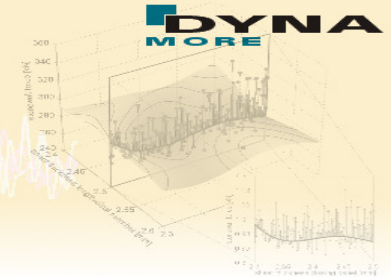


Successive Response Surface Methodology



Methods - Optimization

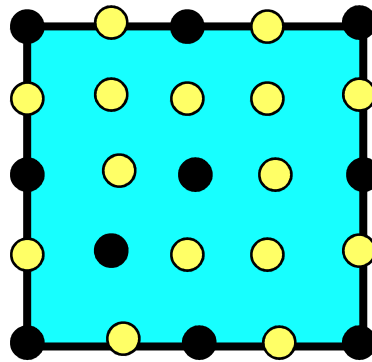
- Introduction/Features
- Methods - Optimization**
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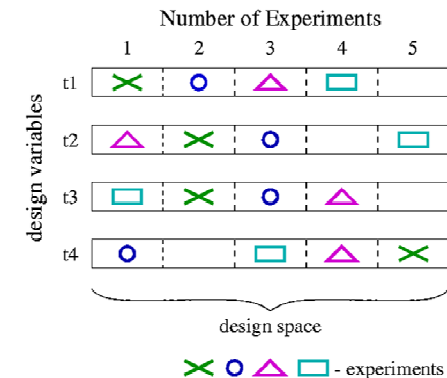
Design of Experiments (DOE) - Sampling Point Selection

- Koshal, Central Composite, Full Factorial
- D-Optimality Criterion** - Gives maximal confidence in the model

$$\max |X^T X|$$

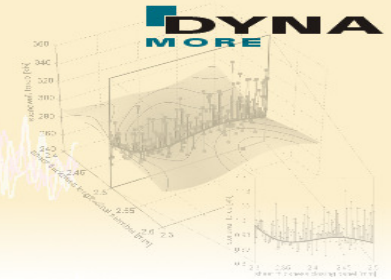


- Monte Carlo Sampling
- Latin Hypercube Sampling (stratified Monte Carlo)
- Space Filling Designs
- User Defined Experiments



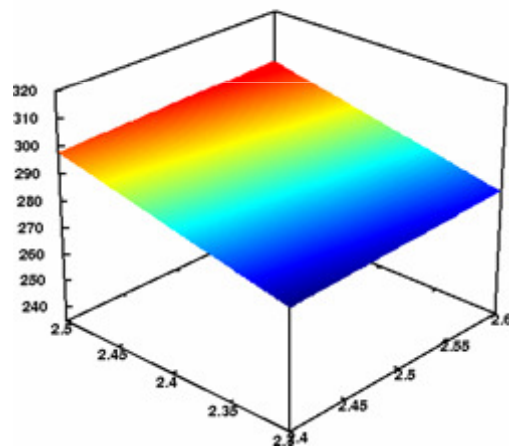
Methods - Optimization

- Introduction/Features
- **Methods – Optimization**
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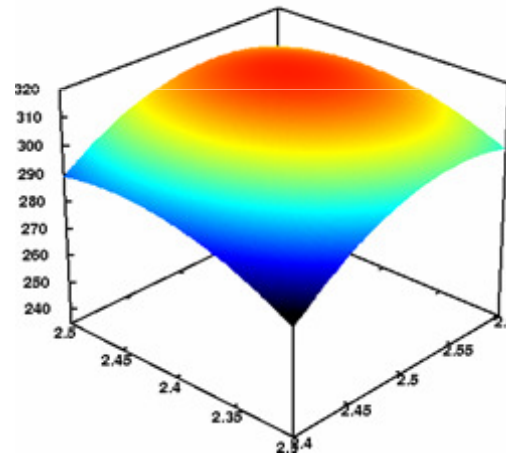


Response Surfaces (Meta Models)

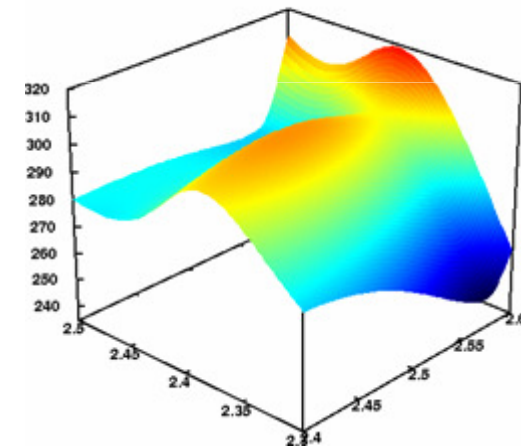
- Linear, Quadratic and Mixed polynomial based
- Radial Basis Functions, Feed Forward Neural Networks and Kriging for global approximations



linear polynomial



quadratic polynomial



neural network

Methodologies – Robustness Investigations

- Introduction/Features
- Methods – Optimization
- **Methods - Robustness**
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DYNA
MORE

Stochastic Analysis - Goals

- Statistical Quantities of Output (Response) due to Variation of Input (Parameter)

- *Mean*
- *Standard deviation*
- *Distribution function*

- Significance of Parameter with respect to Responses

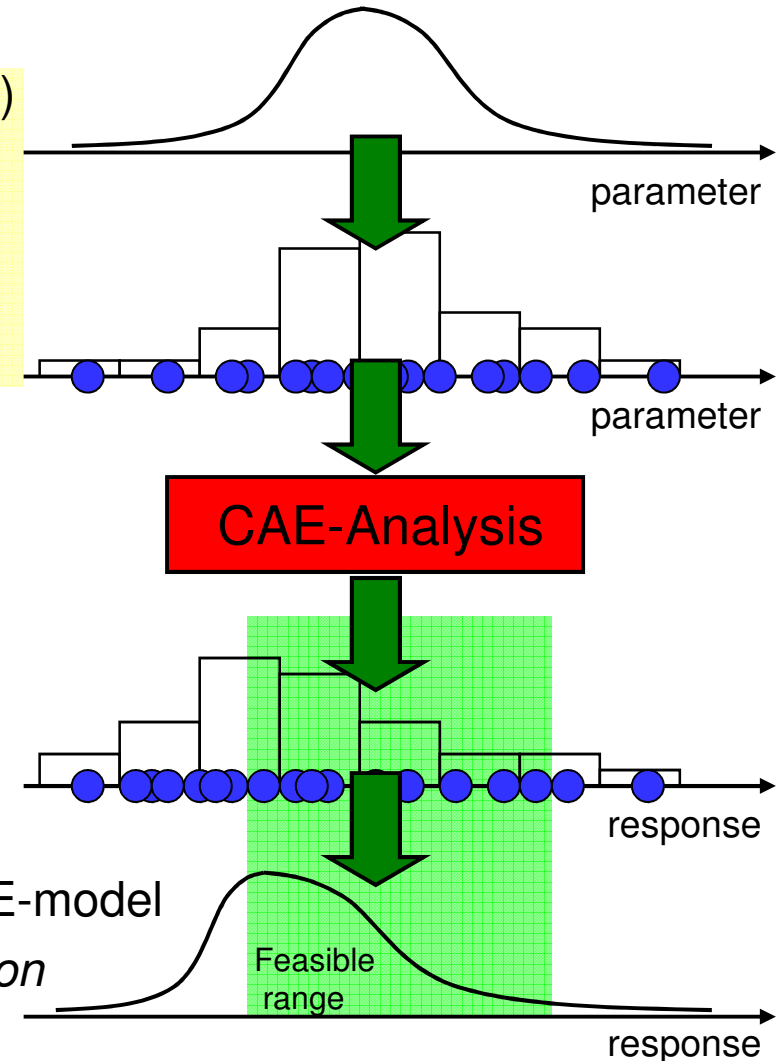
- *Correlation analysis*
- *Stochastic contributions*
- *ANOVA – analysis of variance*

- Reliability Issues

- *Probability of failure*

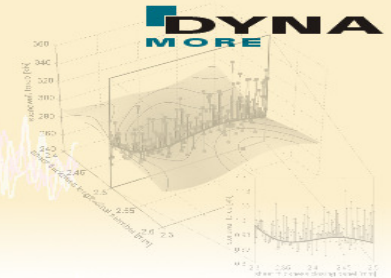
- Visualization of statistical quantities on FE-model

- *Spatial detection of variation/correlation*



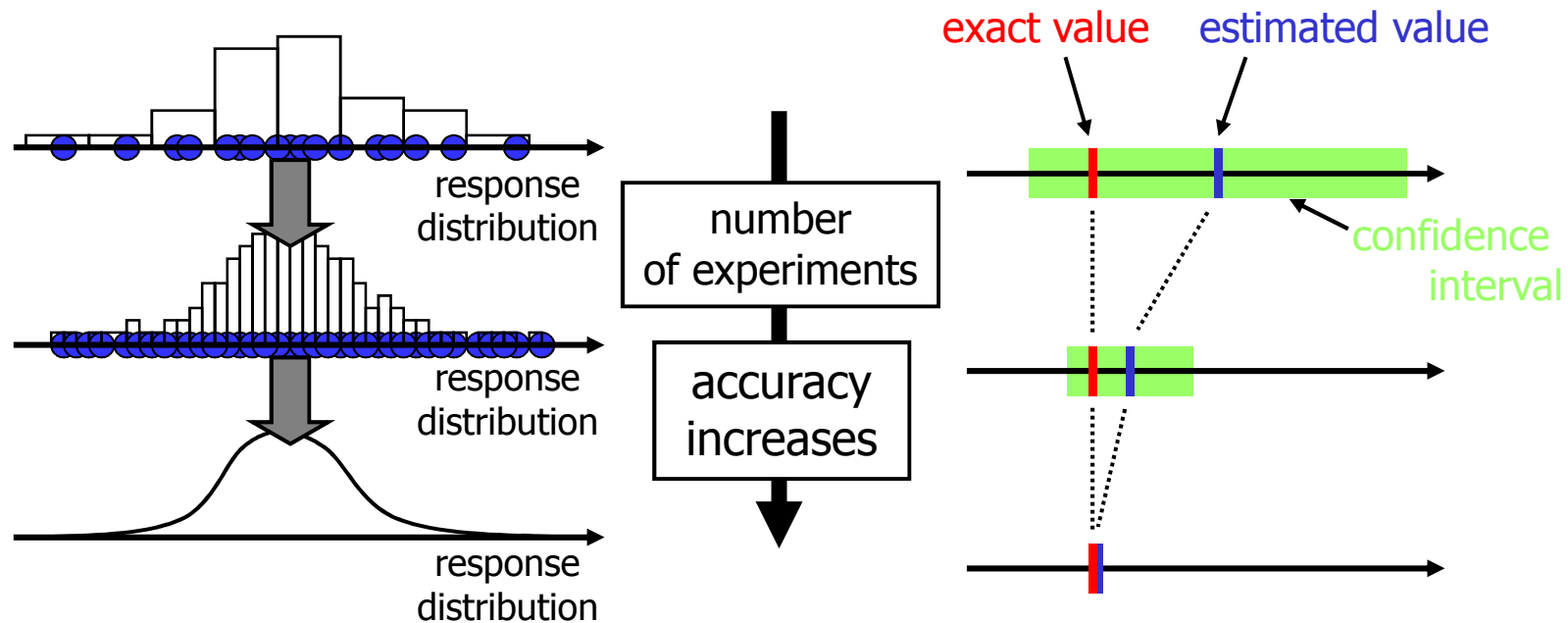
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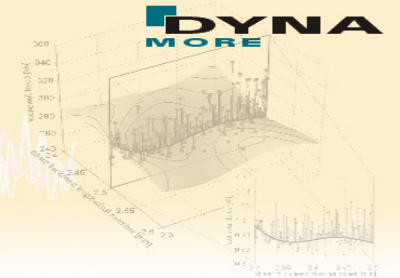
Statistical Quantities of Output due to Variation of Input

- Direct Monte Carlo Sampling
 - *Latin Hypercube sampling*
 - *Large number of FE runs (100+)*
 - *Consideration of confidence intervals for mean, std. dev., correlation coeff.*



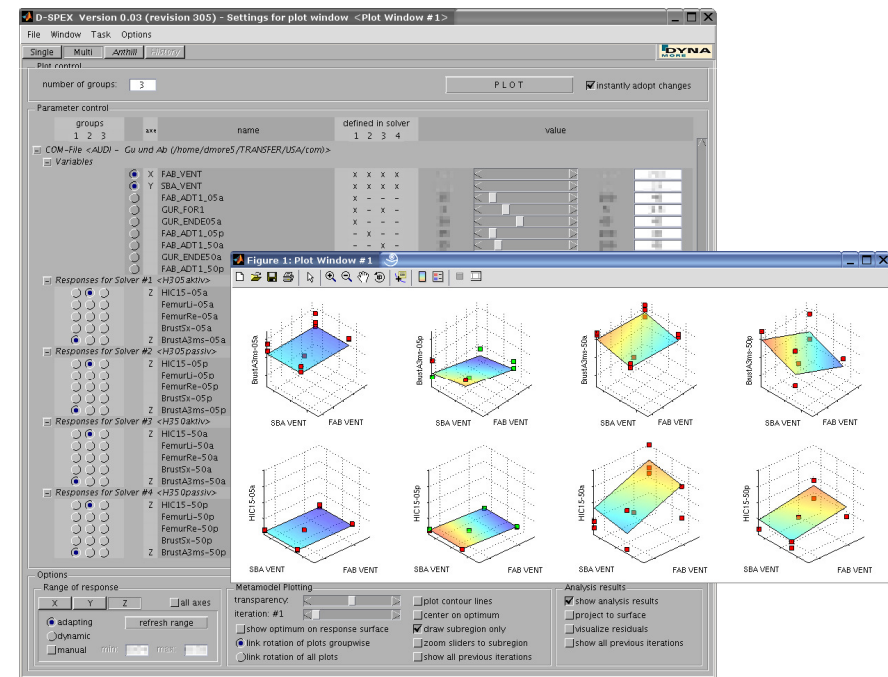
Methodologies – Robustness Investigations

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Statistical Quantities of Output due to Variation of Input

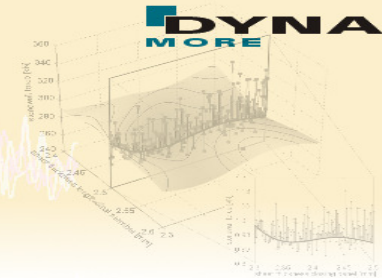
- Monte Carlo using Meta-Models
 - *Response Surface / Neural Network*
 - *Medium number of FE runs (10 – 30+)*
 - *Number of runs depend on the dimension of the problem (number of variables) and the type of the response surface*
 - *Identify design variable contributions clearly*
 - *Exploration of parameter space ->D-SPEX*



Multi Meta-Model exploration with D-SPEX

Methodologies – Robustness Investigations

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- *mean*
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- *distribution function*

- Significance of Parameter with respect to Responses

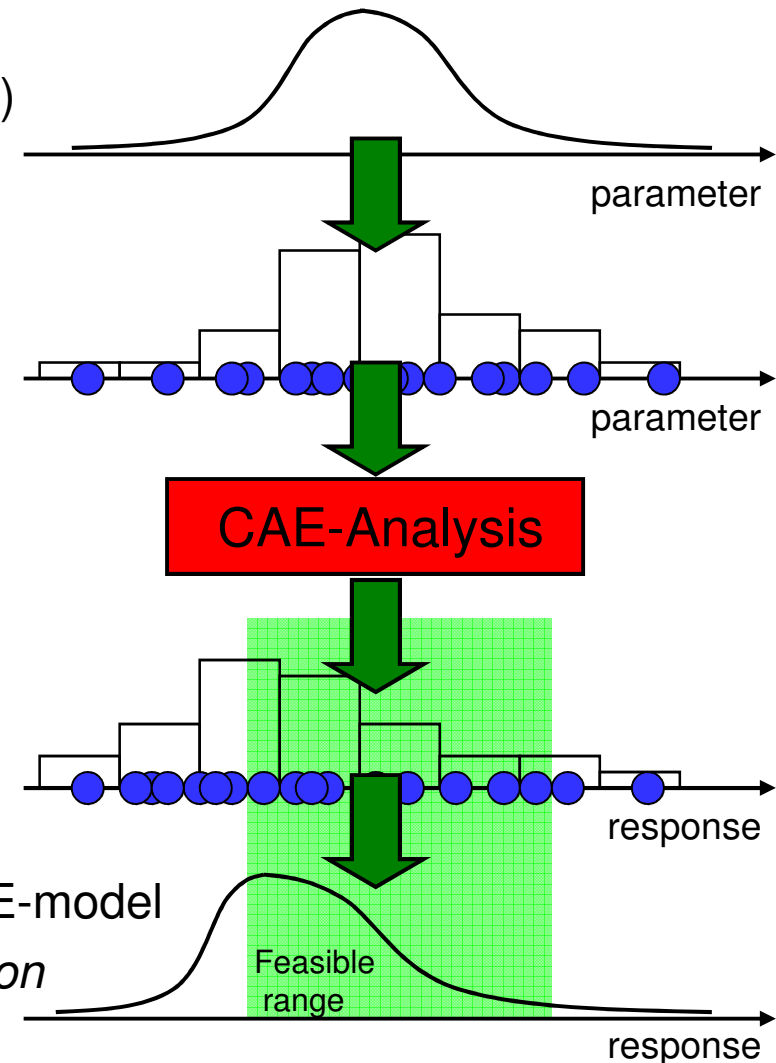
- *correlation analysis*
- *stochastic contributions*
- *ANOVA – analysis of variance*

- Reliability Issues

- *Probability of failure*

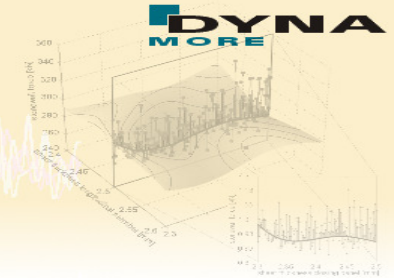
- Visualization of statistical quantities on FE-model

- *Spatial detection of variation/correlation*



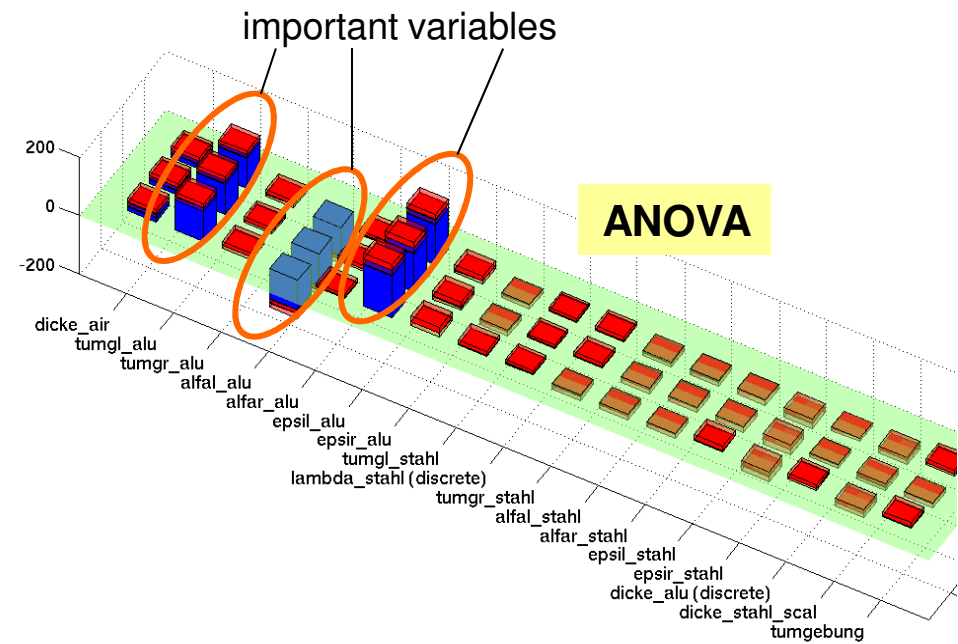
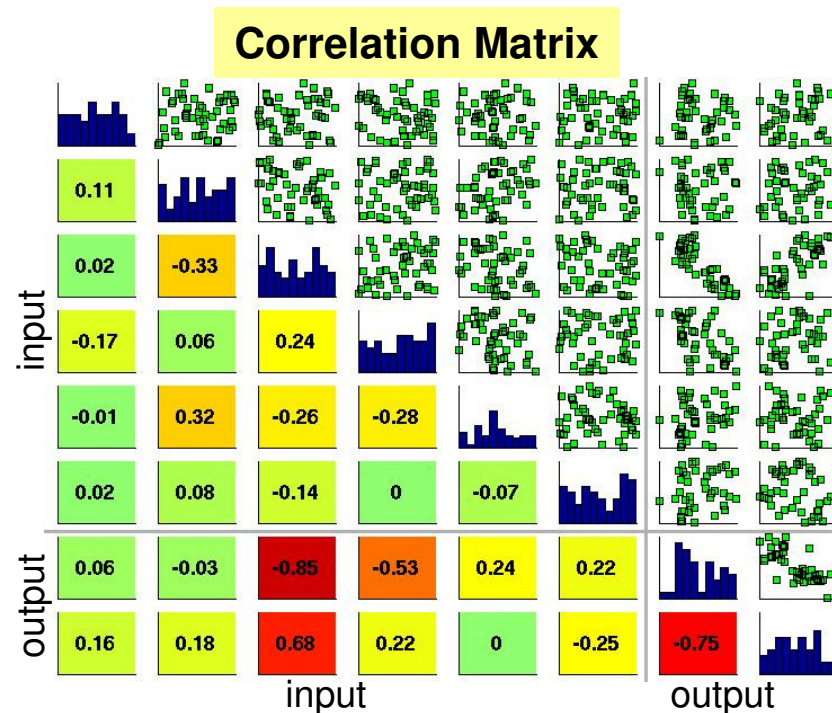
Methodologies – Robustness Investigations

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Significance of Variables

- Correlation Analysis
- ANOVA - Meta-Model based
- Stochastic Contributions – Meta-Model based



Methodologies – Robustness Investigations

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DYNA
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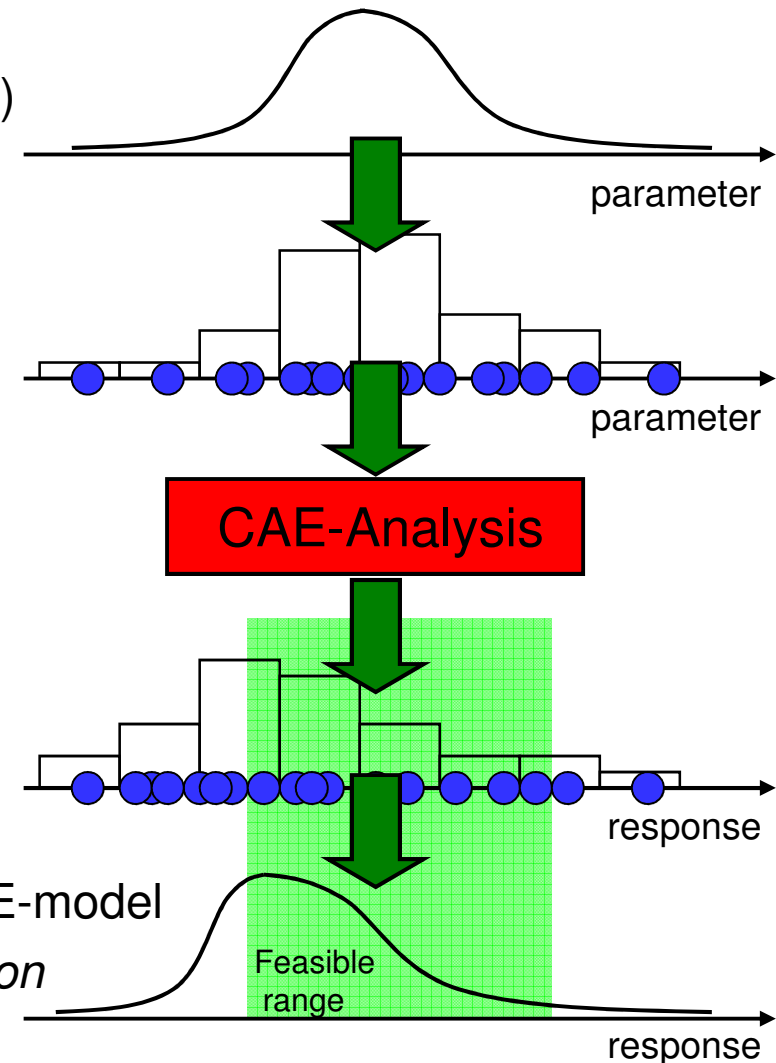
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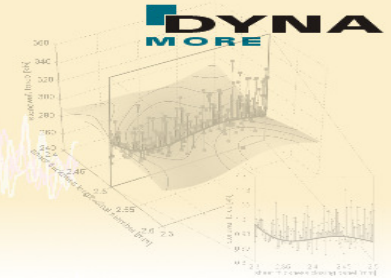
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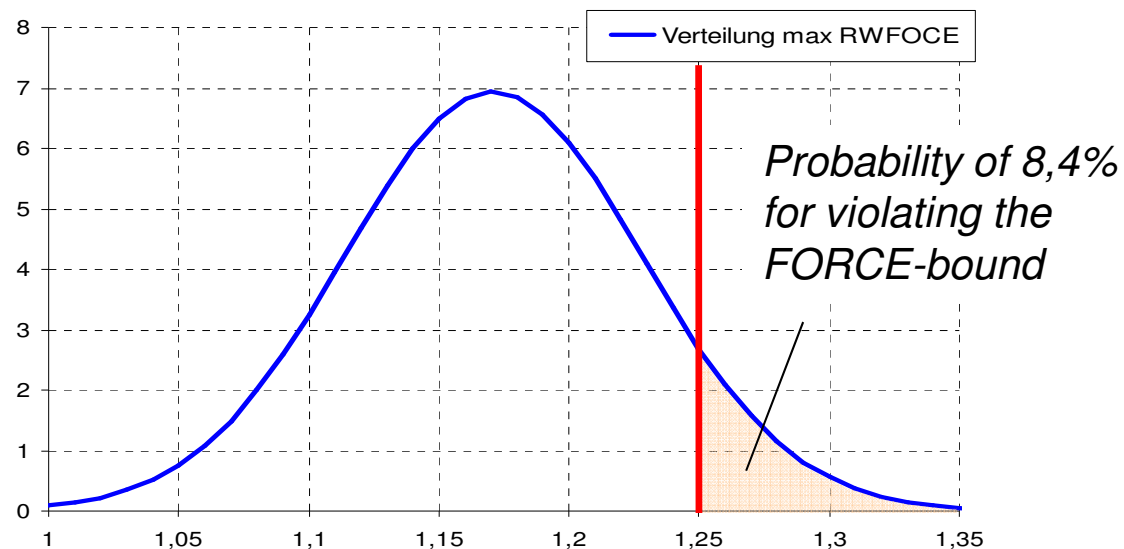
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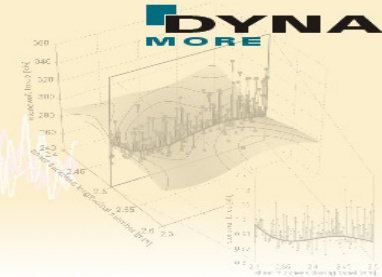
Reliability Analysis

- Probability of failure
- Evaluation of confidence interval
- Prediction error (confidence interval) depends
 - *on the number of runs*
 - *on the probability of event*
 - *not on the dimension of the problem (number of design variables)*



Methodologies – Robustness Investigations

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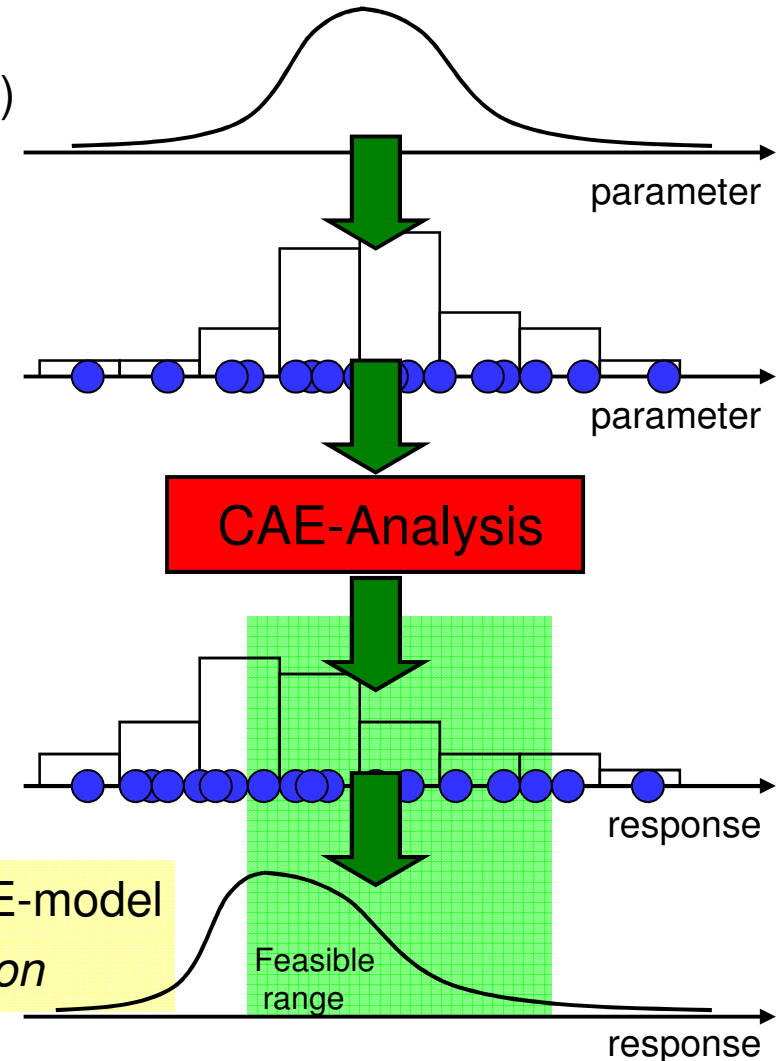
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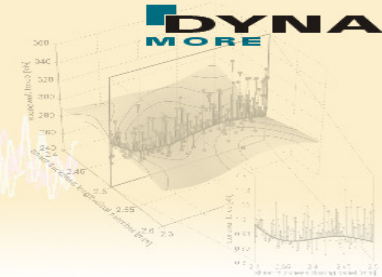
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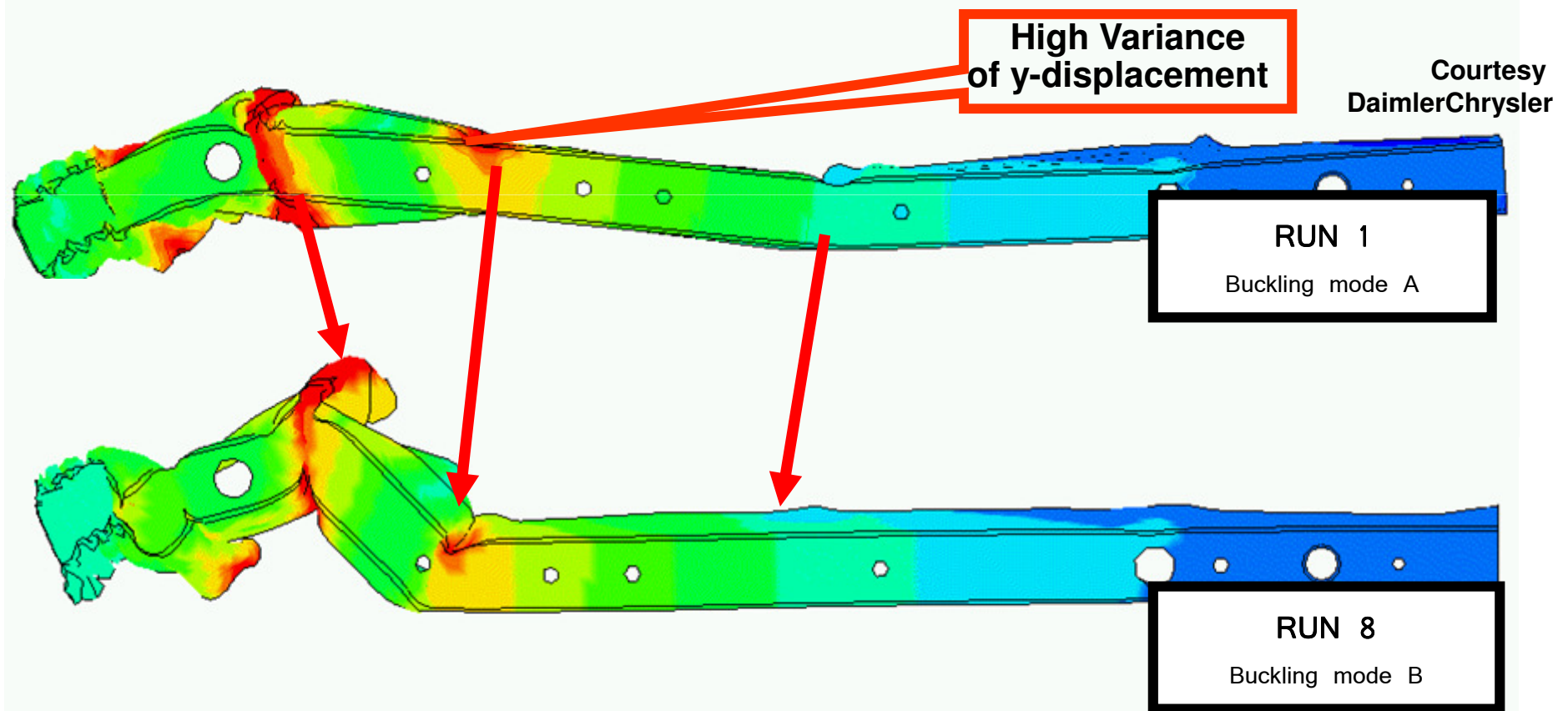
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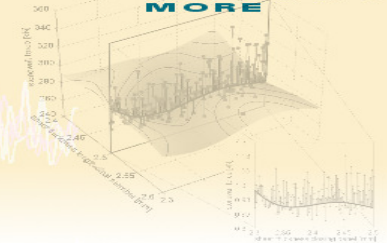
Visualization of Statistical Quantities on FE-model

- Standard deviation of y-displacements of each node (40 runs)



Example I - Optimization

- Introduction/Features
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Optimization of an Adaptive Restraint System



- Four Different Front-Crash Load Cases (FMVSS 208)

Dummy	56 km/h – belted	40 km/h – not belted
Hybrid III 5th Female	H305a (ktiv)	H305p (assiv)
Hybrid III 50th Male	H350a (ktiv)	H350p (assiv)

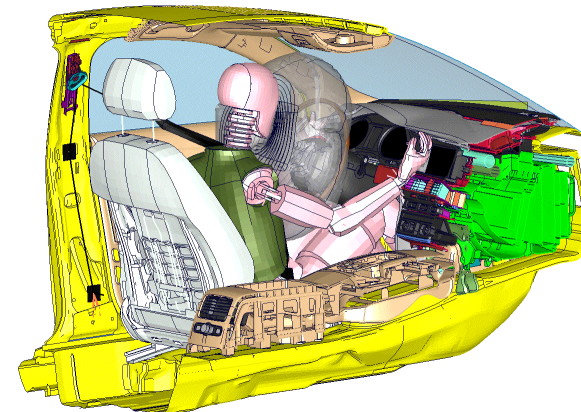
- PAM-Crash Model

- *about 500000 elements*
- *wall clock simulation time ~19 h, 4 cpus, distributed memory*

- Load Case Detection available

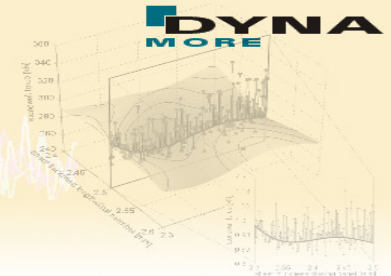
- *Differentiation of the loadcases **belted / not belted** and **“Hybrid III 5th Female“ / „Hybrid III 50th Male“** possible*

- *Trigger time for seatbelt, airbag and steering column might be different*



Example I - Optimization

- Introduction/Features
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Optimization Problem

■ Objective

■ *Minimize Thorax Acceleration*

- > min BrustA3ms-05a
- > min BrustA3ms-50a
- > min BrustA3ms-05p
- > min BrustA3ms-50p

■ Constraints < 80% of regulation requirements

■ *Head Injury Coefficient (15ms)*

- > HIC15-05a
- > HIC15-50a
- > HIC15-05p
- > HIC15-50p

■ *Femur Forces (left/right)*

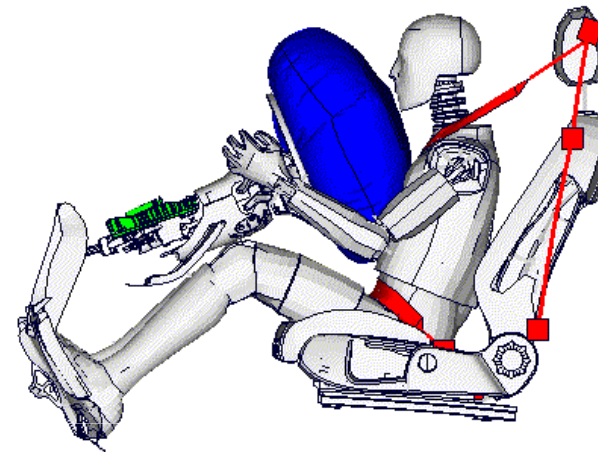
- > FemurLi-05a
- > FemurLi-50a
- > FemurLi-05p
- > FemurLi-50p

■ *Thorax Intrusion*

- > BrustSx-05a
- > BrustSx-50a
- > BrustSx-05p
- > BrustSx-50p

■ *Thorax Acceleration*

- > BrustA3ms-05a
- > BrustA3ms-50a
- > BrustA3ms-05p
- > BrustA3ms-50p

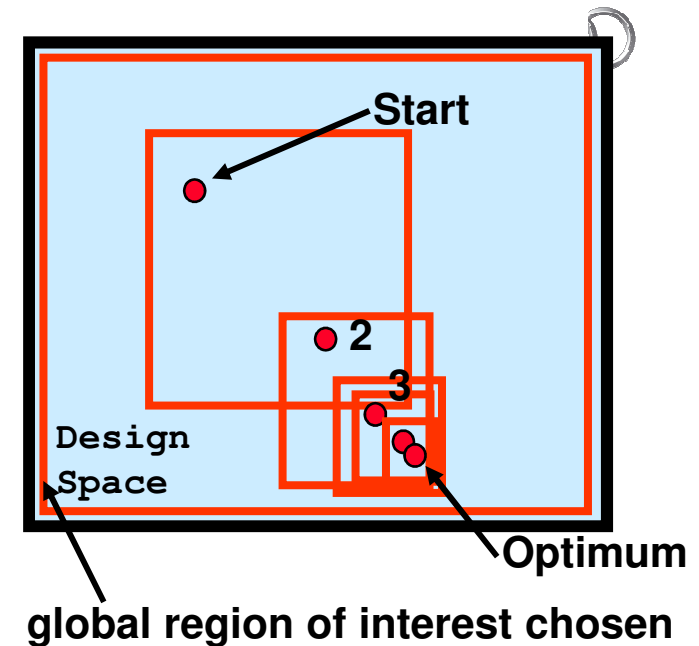


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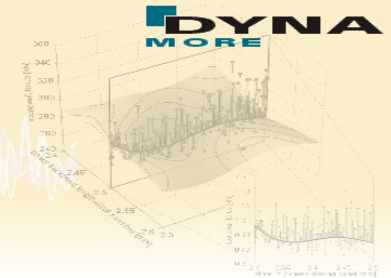
Application of Optimization

- Preferred Configuration at AUDI
 - *Adaptive Restraint System only for Airbag and Seatbelt*
 - *Reduction to 9 Variables in total (active=6, passive=3)*
- LS-OPT Approach: Successive Response Surface Methodology (SRSM) using **linear** polynomial approximations
 - *34 runs per iteration*
 - *D-optimal Design of Experiments (DOE)*
- Results
 - *8 iterations - total runs: 276*
 - *all constraints are fulfilled*
 - *minimization of multi-objective (second step) not applied*

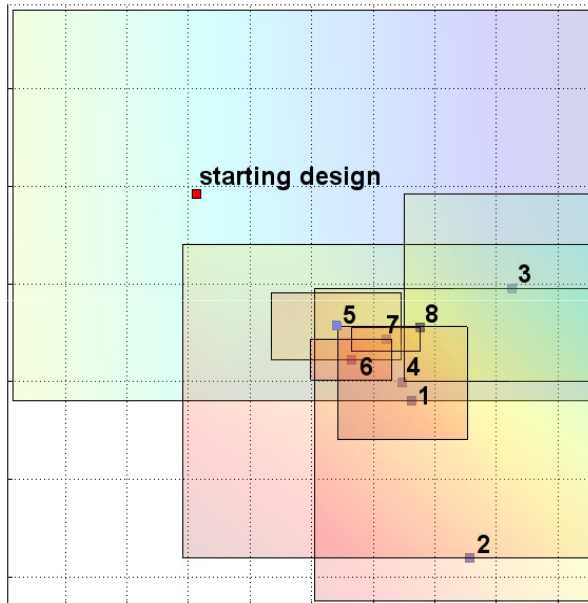


Example I - Optimization

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- Methods – Optimization
- Methods - Robustness
- Examples - Optimization
- Examples - Robustness
- Version 4.1 / Outlook

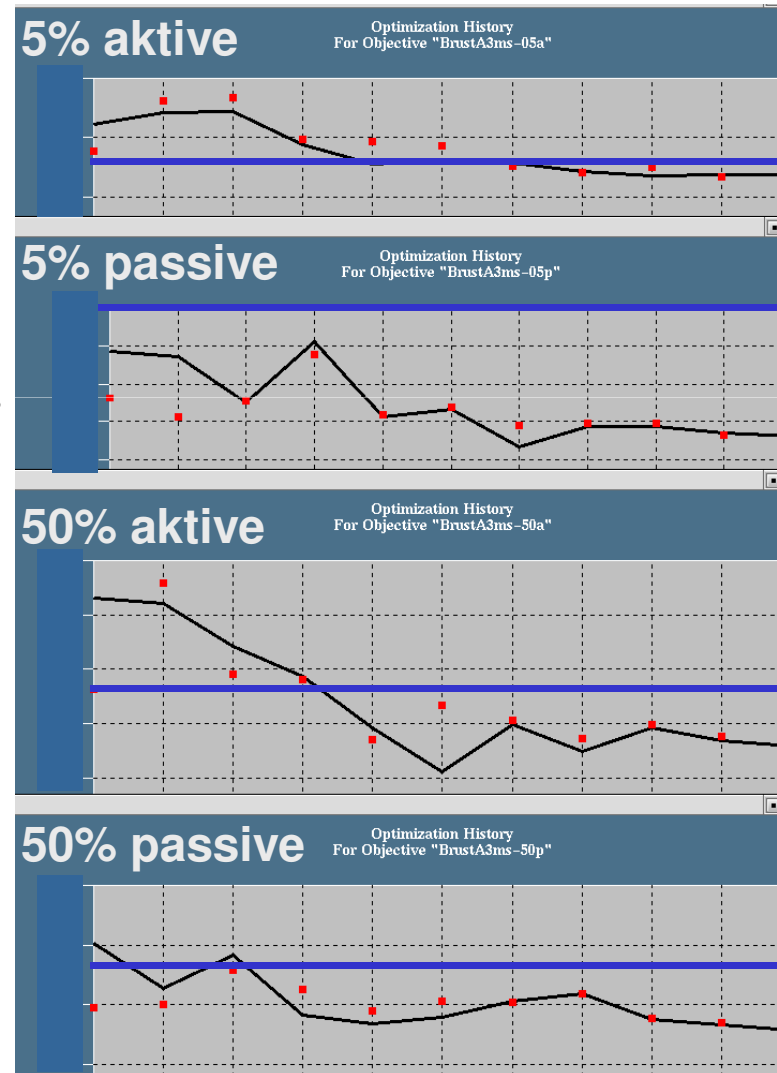


Optimization Progress



a result which meets all requirements is gained in 8 iterations, each with 34 shots

History of Thorax Acceleration



Example I - Optimization

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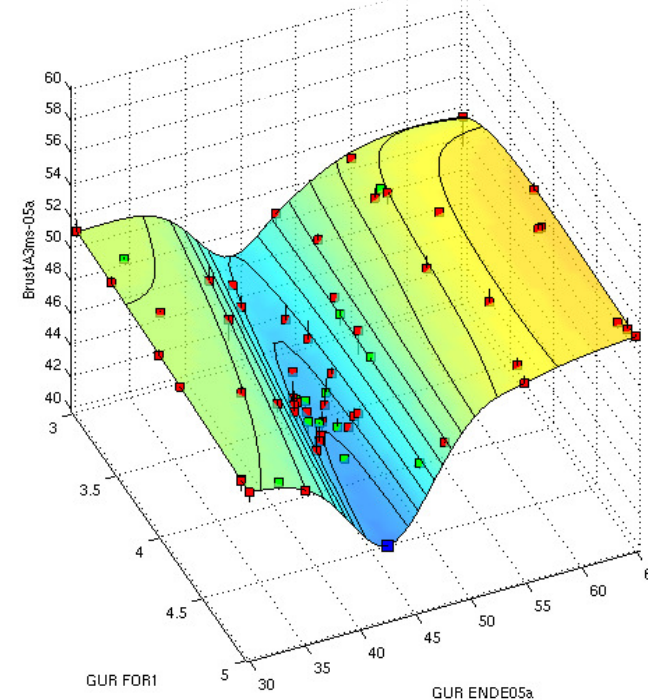
Design Exploring with D-SPEX

The screenshot shows the D-SPEX software interface. The 'Parameter control' panel is visible, listing various variables and their values. A red box highlights the 'Variables' section, and another red box highlights the 'Options' section. A callout box with an arrow points to the slider controls for 'GUR_FOR1' and 'GUR_ENDE05a', with the text 'slider controls for interactive browsing'. Another callout box with an arrow points to the 'Options' section, with the text 'variable and response selection'.

name	value
FAB_VENT	315
SBA_VENT	3.78
FAB_ADT1_05a	35
GUR_FOR1	3
GUR_ENDE05a	30
FAB_ADT1_05p	35
FAB_ADT1_50a	35
GUR_ENDE50a	30
FAB_ADT1_50p	35

Figure 2: Plot Window #1

Trigger Time Seatbelt and Force Limit vs. Thorax Acceleration



variable and response selection

Example I - Optimization

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- **Examples - Optimization**
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DYNA
MORE

Parameter Identification of Plastic Material

- Material properties: nonlinear visco-elastic behaviour
- LS-DYNA hyperelastic/viscoelastic formulation - *MAT_OGDEN_RUBBER (#77)
- Hyperelasticity

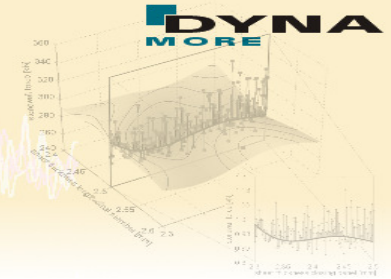
$$W = \sum_{i=1}^3 \sum_{j=1}^n \frac{\mu_j}{\alpha_j} (\lambda_i^{\alpha_j} - 1) + \frac{1}{2} K (J - 1)^2$$

- Prony series representing the viscos-elastic part (Maxwell elements):

$$g(t) = \sum_{m=1}^N G_m e^{-\beta_m t} \quad ; \quad N=1, 2, 3, 4, 5, 6 \quad ; \quad \sigma_{ij} = \int_0^t g_{ijkl}(t - \tau) \frac{\partial \varepsilon_{kl}}{\partial \tau} d\tau$$

Example I - Optimization

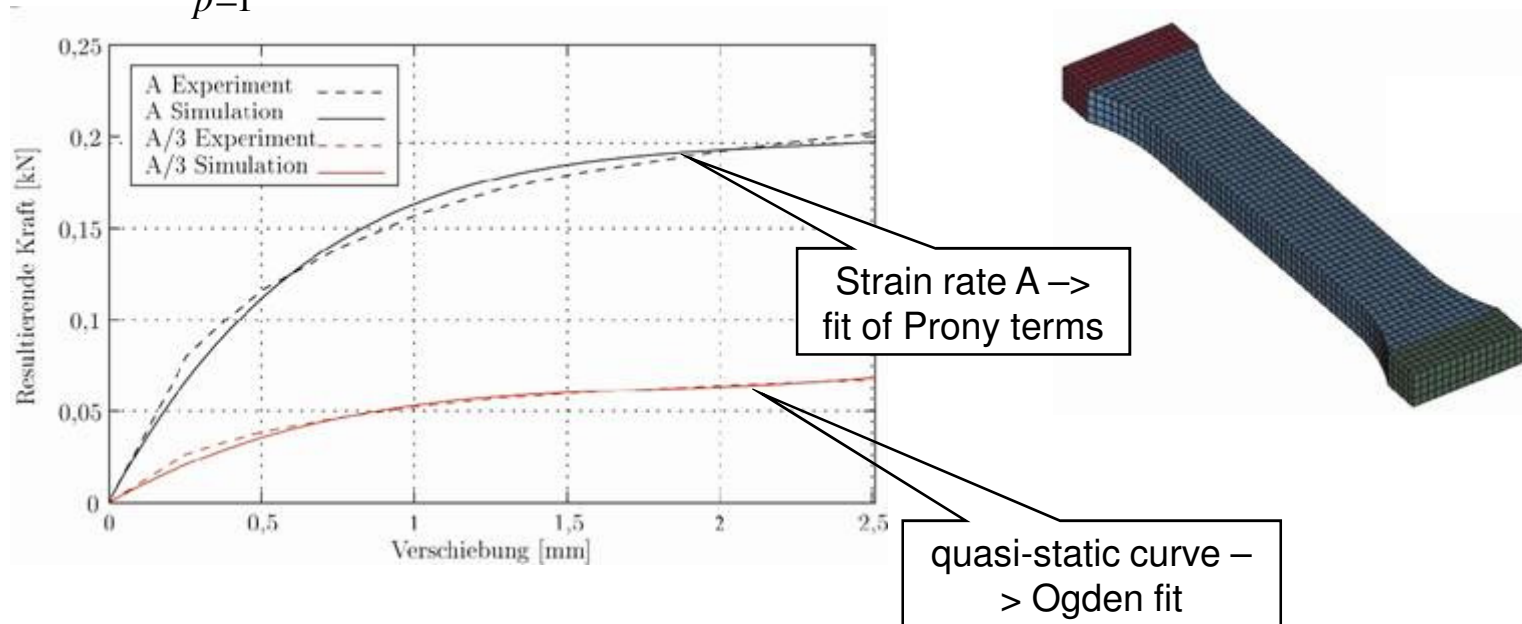
- Introduction/Features
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- Methods - Robustness
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Parameter Identification of Plastic Material

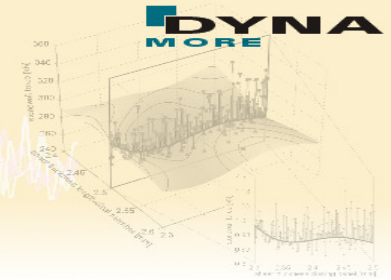
- Minimize the distance between experimental curve and simulation curve
- Least-Squares Objective Function

$$F(\mathbf{x}) = \sum_{p=1}^P \{ [y(\mathbf{x}) - f(\mathbf{x})]^2 \} \rightarrow \min F(\mathbf{x})$$



Example III – Optimization

- Introduction/Features
- Methods – Optimization
- Methods - Robustness
- **Examples - Optimization**
- Examples - Robustness
- Version 4.1 / Outlook

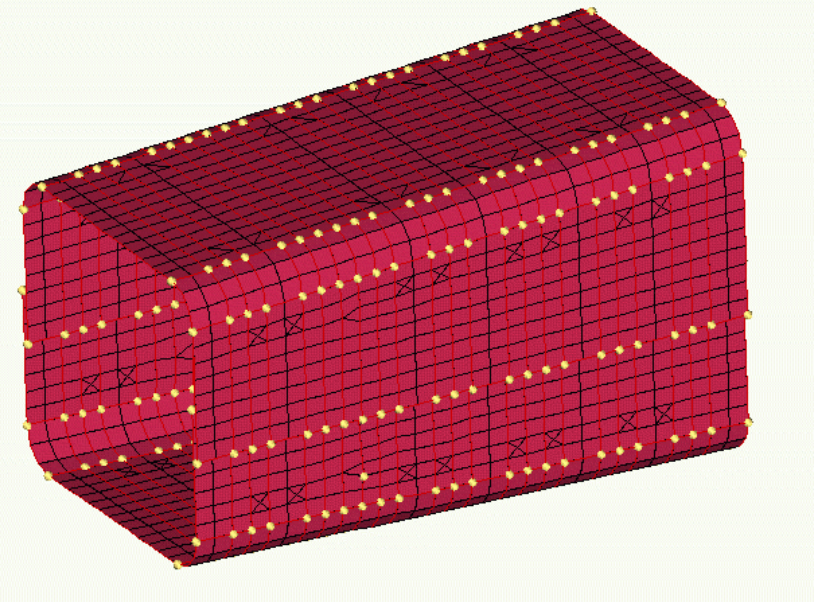
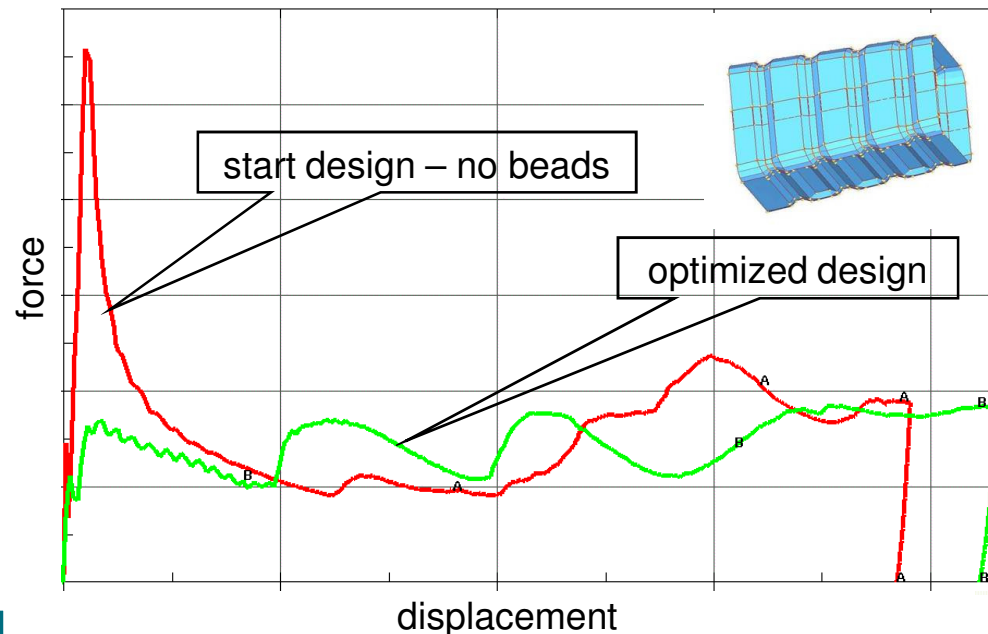


Shape Optimization of a Crash Box

■ Scope of optimization:

- *minimize the maximum crash force*
- *steady-going force progression*

■ Shape variation by using Hypermorph and LS-OPT (20 design variables)



Example I – Robustness

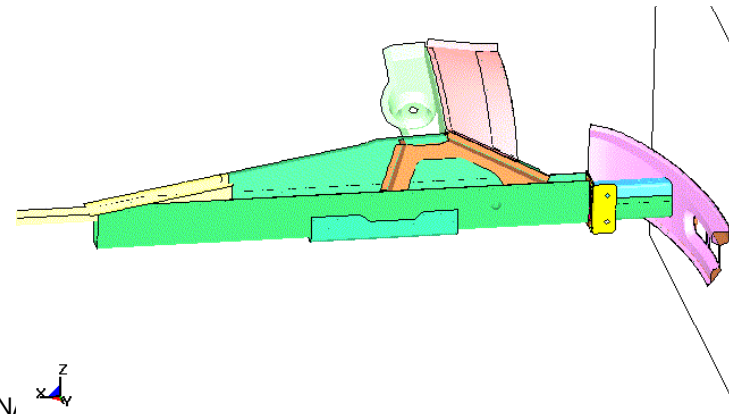
- Introduction/Features
- Methods – Optimization
- Methods - Robustness
- Examples - Optimization
- **Examples - Robustness**
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DYNA
MORE

Robustness Investigations – Monte Carlo Analysis

- Variation of sheet thicknesses and yield stress of significant parts in order to consider uncertainties
- Normal distribution is assumed
 - T_{1134} (Longitudinal Member) $mean = 2.5mm; \sigma = 0.05mm$
 - T_{1139} (Closing Panel) $mean = 2.4mm; \sigma = 0.05mm$
 - T_{1210} (Absorbing Box) $mean = 0.8mm; \sigma = 0.05mm$
 - T_{1221} (Absorbing Box) $mean = 1.0mm; \sigma = 0.05mm$
 - SF_{1134} (Longitudinal Member) $mean = 1.0 ; \sigma = 0.05$
- Monte Carlo analysis using 182 points (Latin Hypercube)



Example I – Robustness

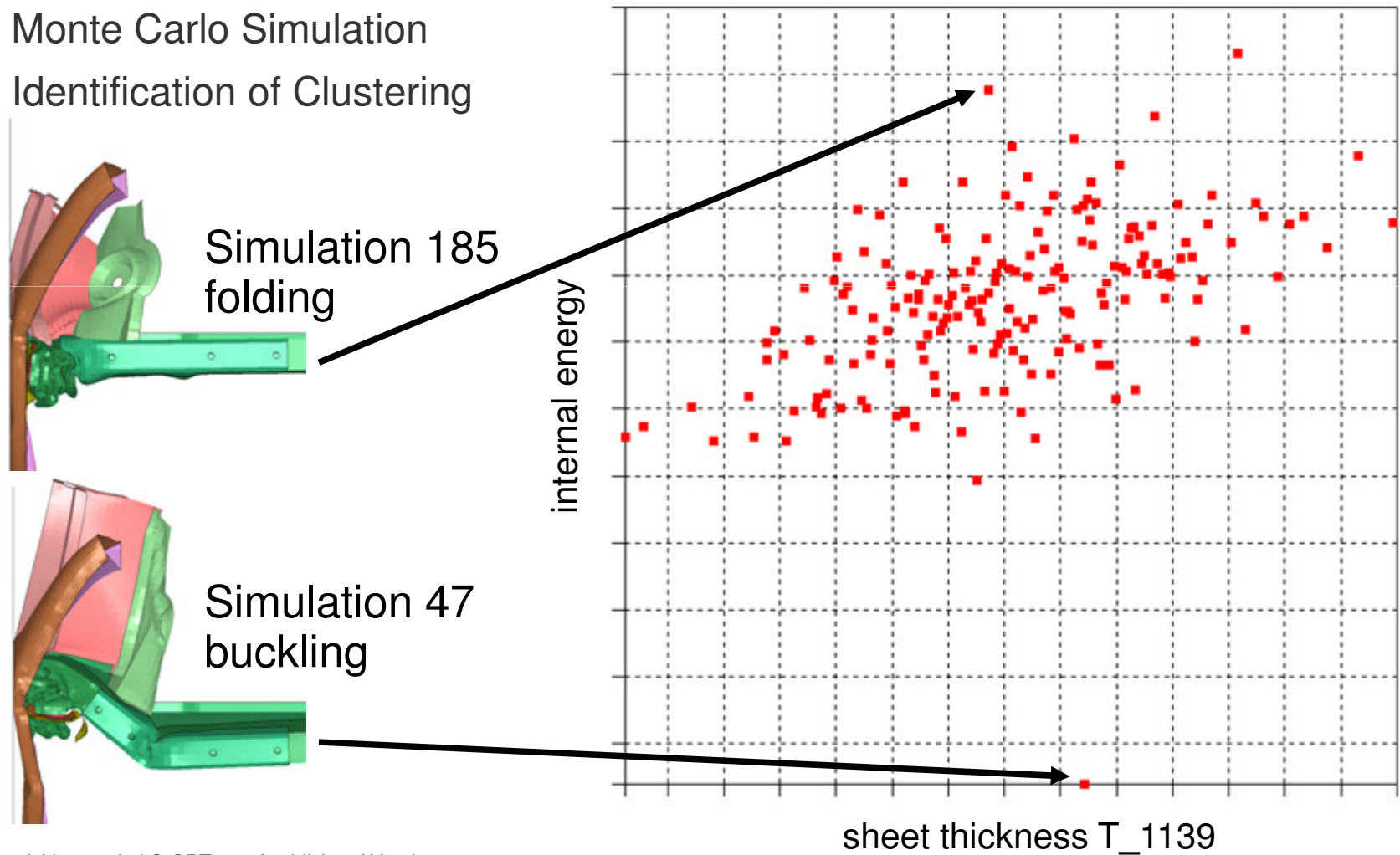
- Introduction/Features
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Tradeoff Plot

- Monte Carlo Simulation
- Identification of Clustering



Example I – Robustness

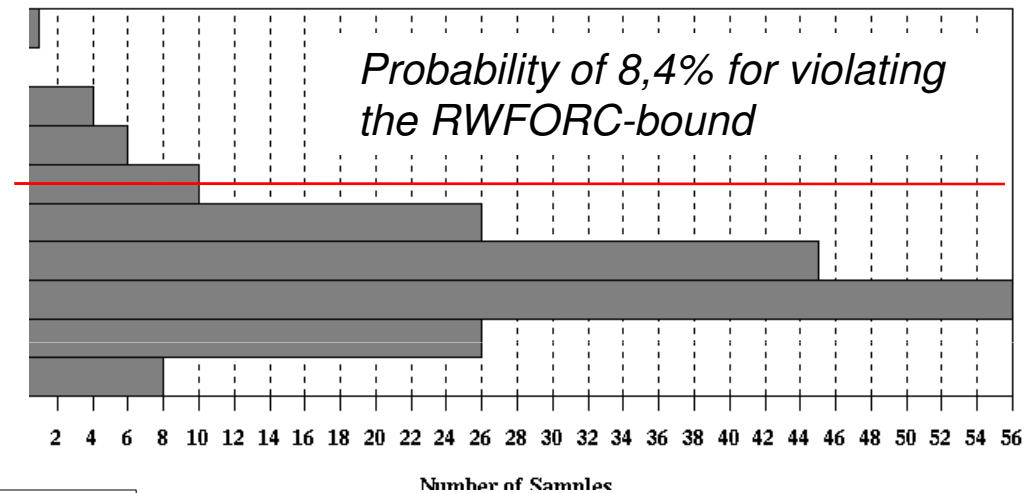
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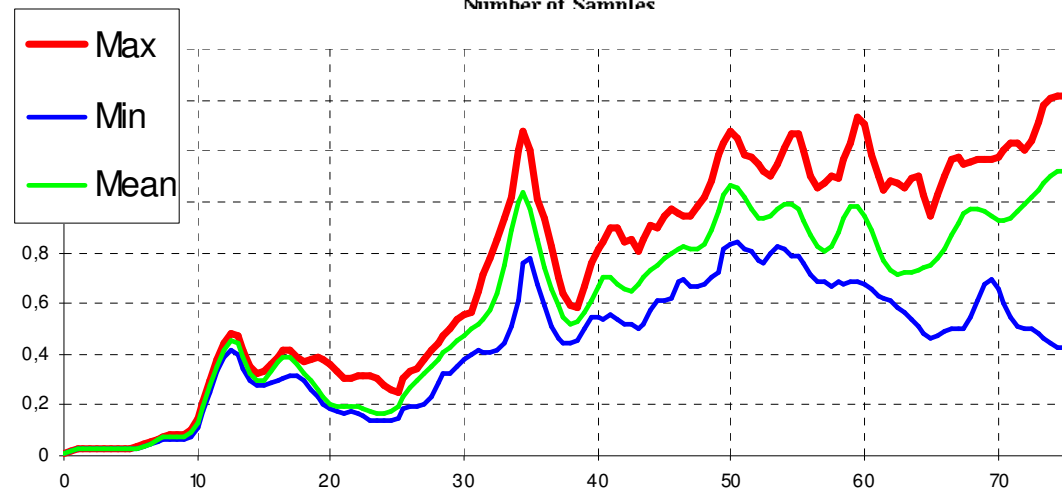
Reliability Analysis

- Histogram of distribution
- Probability of exceeding a constraint-bound



Min-Max Curves

- Plot of minimum, maximum and mean history values
- Gives a confidence interval of history values



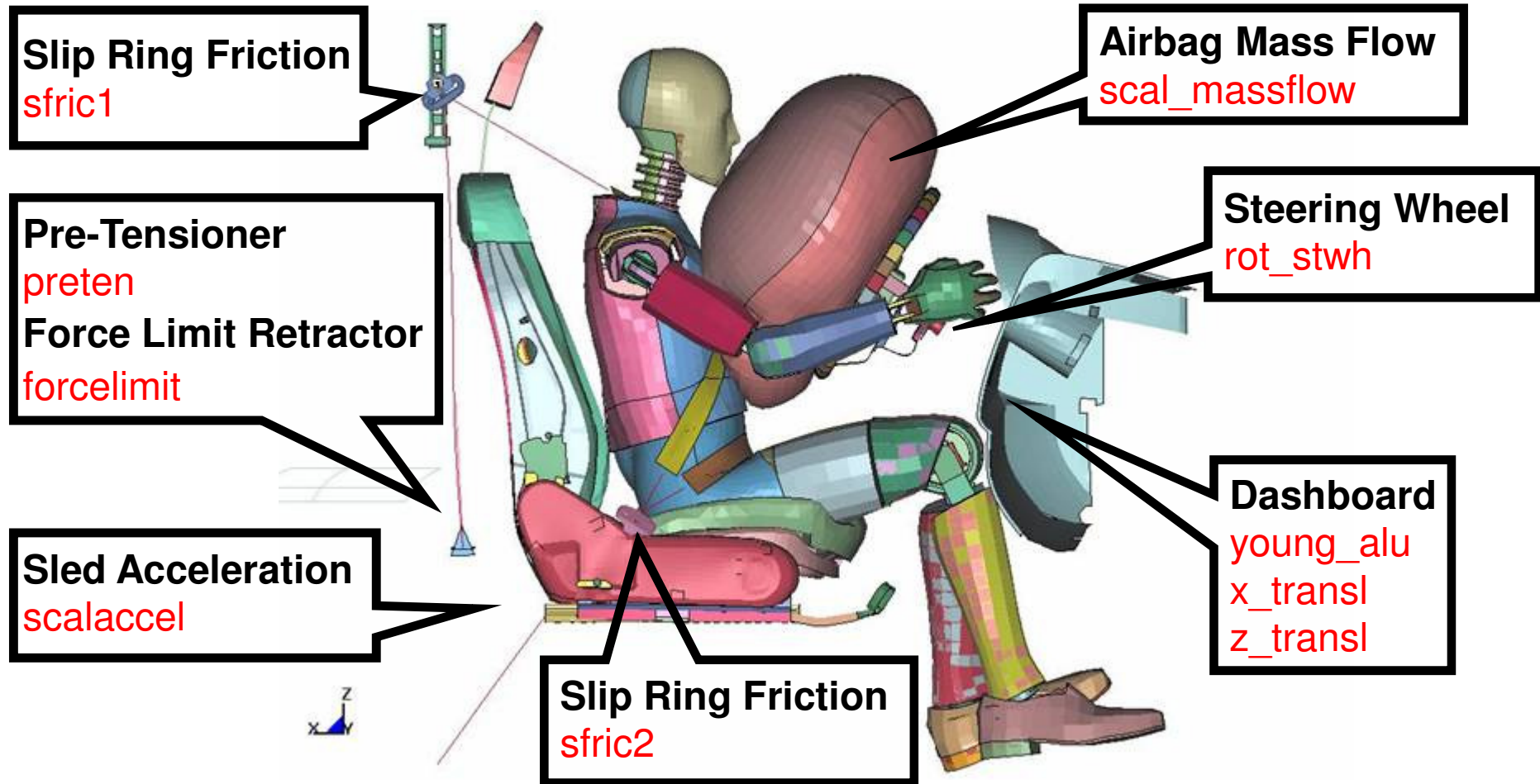
Example II – Robustness

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Design Variables - Uncertainties in Test Set-Up



Example II – Robustness

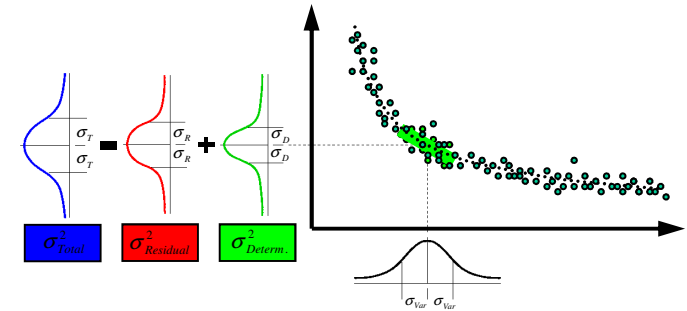
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Stochastic Contribution - Results of 30 Experiments

Design Variable	Standard Deviation of Design Variable	Standard Deviation Contribution					
		HIC36	max_chest_intru	max_b_f_shoulder	max_bf_pelvis	max_chest	max_pelvis
scalaccel	2,5%	3,1%	1,5%	0,1%	2,3%	1,9%	2,9%
sfri1	25,0%	1,3%	0,6%	4,1%	1,8%	0,7%	0,7%
sfri2	25,0%	0,5%	0,6%	0,1%	3,7%	0,1%	0,1%
preten	4,4%	0,0%	0,5%	0,0%	1,1%	0,3%	0,2%
forcelimit	5,6%	1,3%	0,4%	4,4%	0,6%	1,4%	0,2%
rot_stwh	4,8%	0,5%	0,1%	0,1%	0,0%	0,1%	0,1%
transl_x	50,0%	0,1%	0,1%	0,7%	4,5%	0,5%	0,8%
transl_z	50,0%	1,2%	1,0%	0,3%	1,6%	0,2%	0,9%
scalmassflow	5,0%	1,8%	1,8%	0,6%	2,2%	0,6%	0,9%
young_alu	5,0%	0,3%	0,3%	0,0%	0,5%	0,1%	0,1%
all variables		4,3%	2,8%	6,1%	7,2%	2,6%	3,4%
residuals		4,7%	1,9%	1,8%	6,0%	3,5%	2,3%
Total		6,4%	3,4%	6,3%	9,4%	4,3%	4,1%



Contribution of variation of design variables to variation of results

Meta-model space

Residual space

Total Variation

Example II – Robustness

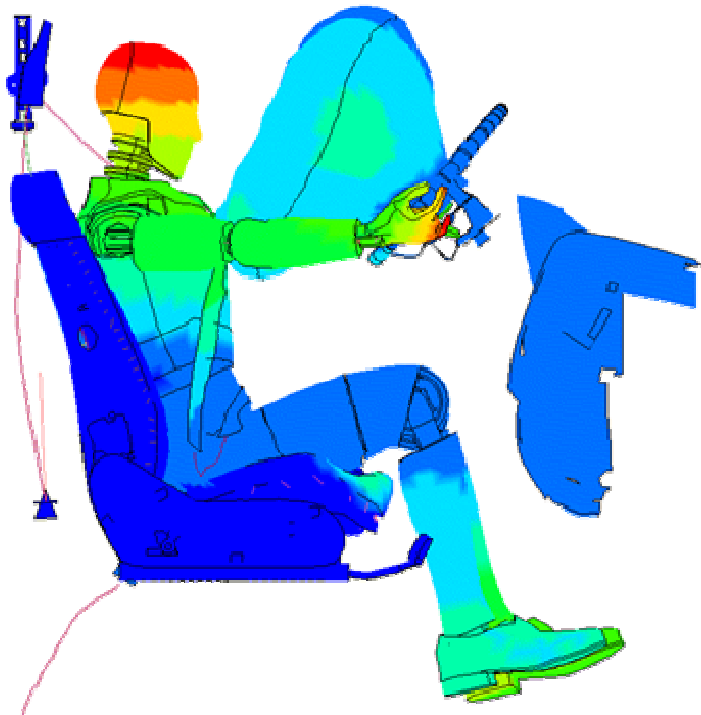
- Introduction/Features
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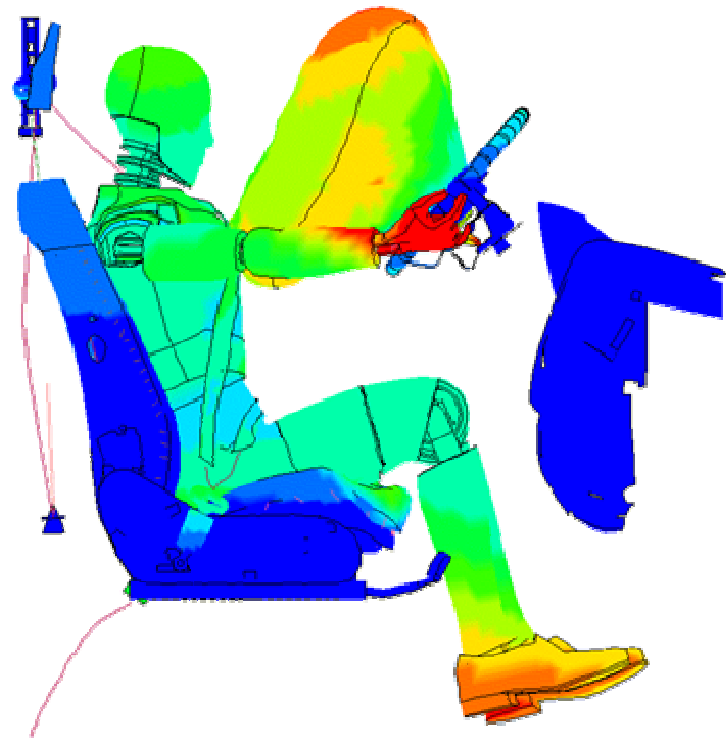
DYNA
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Standard deviation of x-displacements of each node (120 runs)

(a) Deterministic (Meta-Model)



(b) Residual (Outliers)



Outlook – Version 4.1

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➔ Outlook – Version 4.1

■ LS-OPT/Topology (available end 2009)

- *Nonlinear topology optimization*

- *LS-DYNA based*

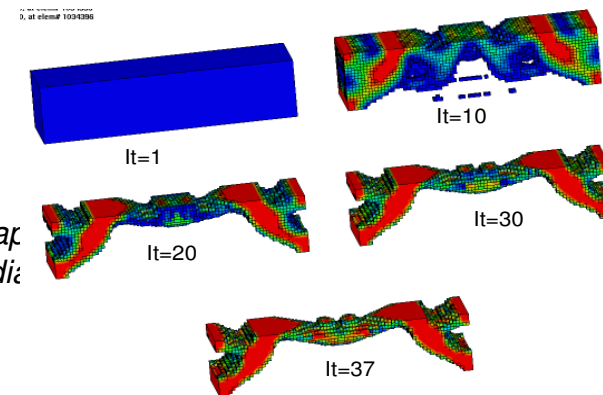
- *Multiple load cases*

- *Linear as well as non-linear*

- *Design part selection*

- *Methodology: Hybrid Cellular Automata*

*Ref.: Hybrid cellular automata with local control rules: a new approach
Quevedo W., Patel N., Renaud J. (University Notre-Dame, Indiana)
Structural and Multidisciplinary Optimization, 2005, Brazil*



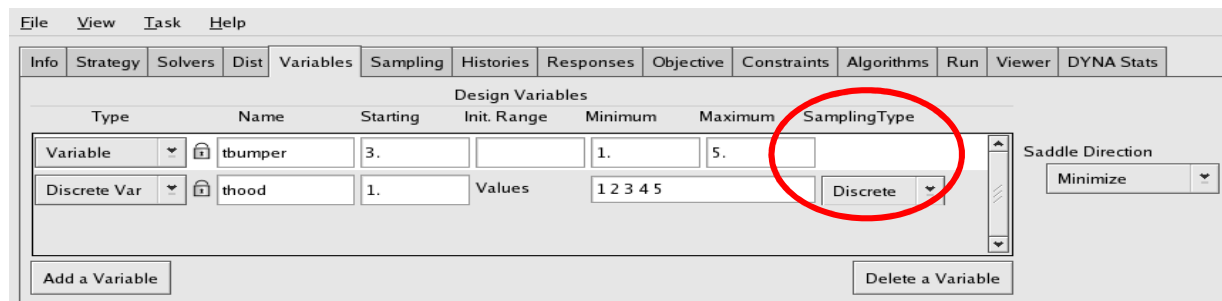
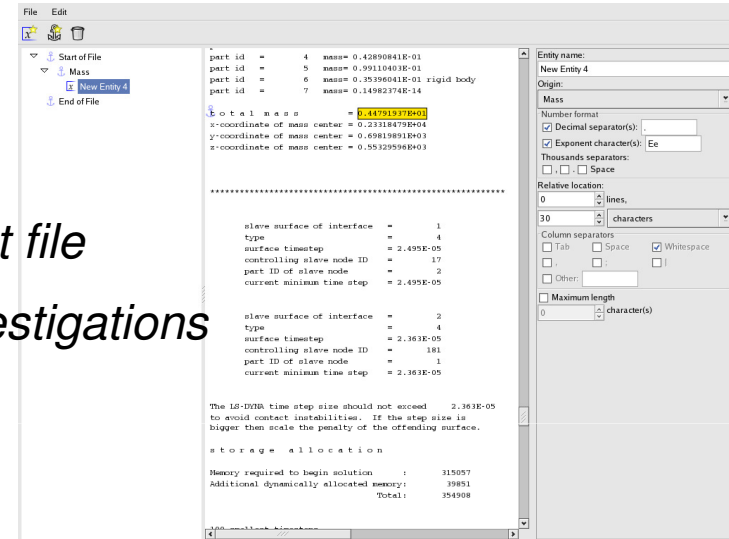
Outlook – Version 4.1

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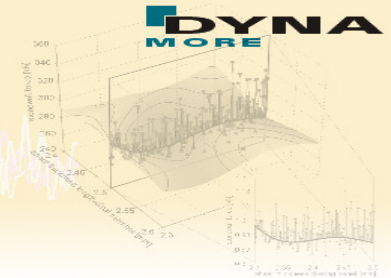
➔ Outlook – Version 4.1

- Generic File extractor
 - Extraction of values from any ASCII input file
- Correlated Input Variables for Stochastic Investigations
- Additional injury criteria (DYNA extraction)
 - IHS, neck/tibia indices,...
- Full Factorial Discrete
 - Schedule all combinations of discrete variables
- Discrete sampling on a variable basis



Outlook – Version 4.1

- Introduction/Features
- Methods – Optimization
- Methods - Robustness
- Examples - Optimization
- Examples - Robustness
- Outlook - Version 4.1



➔ Outlook – Version 4.1

- Improved Space Filling algorithm
 - *Greater accuracy: more boundary points*
- Frequency/Mode Tracking
 - Available for NASTRAN
 - MAC and orthogonality criterion
- Global Sensitivity Analysis with Sobol Indices (GSA)
- Pareto Optimal Solutions
 - *Self-Organizing Maps (Ver. 4.1)*
- Many new Features within the “viewer”
 - *e.g. history curves visualization*

$$\max_i [(\boldsymbol{\varphi}_r^T \mathbf{M}_r) \boldsymbol{\varphi}_i]$$

