# LS-DYNA Beam Elements: Default and User Defined Cross Section Integration

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## ABSTRACT

LS-DYNA provides several beam element formulations, see the keyword description for *Section_Beam in the User's Manual. Several of these beam element formulations support user supplied integration of the cross section, via the *Integration_Beam keyword. While most LS-DYNA users are familiar with the similar through-the-thickness integration algorithm for shell elements, which is made trivial by the rectangular cross section geometry assumed for shell elements, the numerical integration of even simple beam element cross sections requires more effort, and as will be demonstrated, more planning.

In this article, a detailed explanation of the beam element cross section integration algorithm is presented. Simple suggestions for calculating, and checking, user provided integration rules are illustrated through several examples. The examples also provide suggestions for improving the LS-DYNA Standard Cross Section Types, available via the ICST parameter of the *Integration_Beam keyword.

## ANALYTICAL INTEGRATION

To provide a basis for assessing the numerical integration algorithm, used in LS-DYNA, for beam element cross sections, we first establish the corresponding analytical results. There are two cross section integrations that will be considered: axial force and bending moment; the former is relatively straight forward and is included only for completeness.

### Axial Force

The axial force in a beam element is obtained by integrating the axial stress over the cross section,

$$F = \int \sigma dA = \int \sigma\left(y,z\right) dydz \tag{1}$$

where $\sigma = \sigma\left(y,z\right)$ is the axial stress which typically varies over the cross sectional area in the $y-z$ plane.

### Bending Moment

The moments, about the local $y$ and $z$ axes, in a beam element are obtained by integrating the axial stress times the appropriate distance (moment arm) over the cross section,

$$\begin{aligned} M_y &= \int \sigma z dA = \int \sigma\left(y,z\right) z dydz \\ M_z &= \int \sigma y dA = \int \sigma\left(y,z\right) y dydz \end{aligned} \tag{2}$$

where local $y$ and $z$ axes are often the centrodial axes of the cross section, but may be any convenient cross sectional axes. Recall that the centrodial axis can be located relative to any axis via

$$z_c = \frac{\int zdA}{A} \tag{3}$$

where $z_c$ provides the distance from the $z$ axis to the centroid. Centroids lie along planes of symmetry, if they exist in a cross section, as the above integral is zero for symmetric limits of the variable $z$ , i.e. the cross section is symmetric with respect to the $x-y$ plane.

For the special case of pure bending about only one axis, say the $y$ axis, with a symmetric cross section, the axial stress may be written as

$$\sigma = \sigma\left(z\right) = \sigma_0 \frac{2z}{h} \tag{4}$$

where $\sigma_0$ is the maximum (outer fiber) stress and $-h/2 < z < h/2$ where $h$ is the height of the cross section, see **Figure 1**, and for simplicity consider a rectangular cross section of width $b$ .

Substitution of the above linear stress distribution into the first of Equations (2) yields

$$\begin{aligned} M_y &= \int_A \sigma zdA = \int_{-b/2}^{b/2} dy \int_{-h/2}^{h/2} \sigma\left(z\right) zdz = \sigma_0 \frac{2b}{h} \int_{-h/2}^{h/2} z^2 dz \\ &= \sigma_0 \left(\frac{2}{h}\right) \frac{bh^3}{12} = \frac{\sigma_0}{c} I_{yy} \end{aligned} \tag{5}$$

Inverting the above provides the familiar strength of materials result

$$\sigma_0 = \frac{Mc}{I} \tag{6}$$

where $c = h/2$ and $I_{yy}$ is the cross sectional moment of inertia, and has the following familiar form

$$I_{yy} = \int_A z^2 dA = b \int_{-h/2}^{h/2} z^2 dz = \frac{bh^3}{12} \tag{7}$$

when $z$ is the centrodial axis.

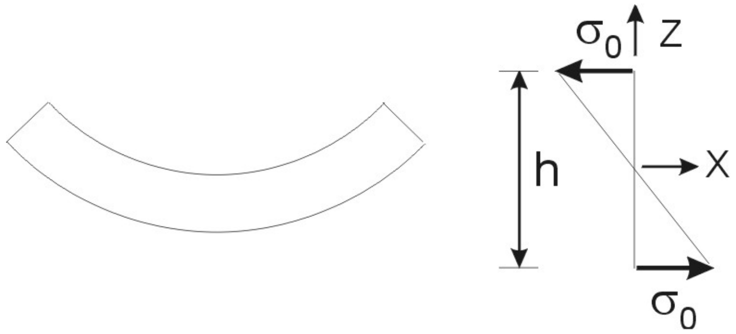


**Figure 1** Pure bending stress distribution along the cross section height.

## PARALLEL-AXIS THEOREM

The above integration, Equation (7), to obtain the cross sectional moment of inertia, for more complex beam cross sections, can become tedious. Fortunately, the Parallel-Axis Theorem provides a relatively straightforward technique for dividing complex cross sections into simple shapes, especially rectangles, to calculate the cross sectional moment of inertia. Recall the Parallel-Axis Theorem:

$$I = \overline{I} + Ad^2 \tag{8}$$

This formula expresses that the moment of inertia, $I$, of an area with respect to any given axis, $z$, is equal to the moment of inertia, $\overline{I}$, of the area with respect to its (local) centrodial axis, $\overline{z}$, parallel to $z$, plus the product, $Ad^2$, of the area, $A$, and the square of the distance, $d$, between the two axes.

Use will be made of the Parallel-Axis Theorem in assessing the numerical integration of beam cross sections.

## NUMERICAL INTEGRATION

The basis of the cross section numerical integration is to divide the cross section into simple rectangular regions, see for example **Figure 2**. The center of each region is an integration point. First the strain is evaluated at each integration point, based on the curvature (rotations) and relative nodal displacements. Then using the constitutive relation, the stresses corresponding to the strains are evaluated at each integration point. Finally, the stresses are integrated numerically to produce the axial force (thrust) and moments.

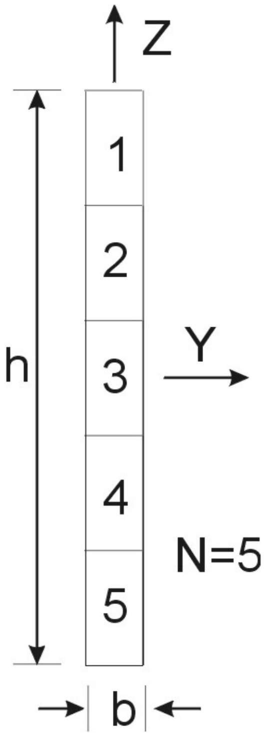


**Figure 2** Rectangular cross section divided into 5 equal regions for numerical integration.

### Axial Force Calculation

The axial force integration, given by Equation (1), is approximated as

$$F = \int \sigma dA \approx \sum \sigma_i A_i \tag{9}$$

where the sum is taken over all the integration points and, $A_i$, is the rectangular area around the integration point where the stress, $\sigma_i$, is evaluated. Provided the cross section can be represented by rectangular regions, the numerical approximation given by Equation (9) is usually quite good, and is exact for linear elastic beams, and shells, even for $i = 1$.

### Bending Moment Calculation

The bending moment integration, given by Equations (2), is approximated in a manner similar to the axial force as

$$M_y = \int \sigma z dA \approx \sum \sigma_i z_i A_i \tag{10}$$

where the distance $z_i$ is measured from the integration point to the $y$ axis, about which the moment is being calculated.

To assess the accuracy of the above approximation, consider the pure bending linear stress distribution given previously by Equation (4), then the numerical approximation may be written as:

$$M_y \approx \sum \sigma_i z_i A_i = \frac{2\sigma_0}{h} \sum z_i^2 A_i \tag{11}$$

A comparison with the corresponding analytical result, Equation (5), indicates that the approximation depends on the ability of the numerical sum, in Equation (11), to represent the cross section moment of inertia $I_{yy}$, i.e.

$$I_{yy} \approx \sum z_i^2 A_i \tag{12}$$

The Parallel-Axis Theorem can now be used to investigate the above approximation. Rewriting Equation (8) in the notation of Equation (12), yields

$$I_{yy} = \overline{I}_{yy} + z^2 A \tag{13}$$

If we now consider the same simple rectangular cross section, of width $b$ and height $h$, but subdivide the cross section along the height direction into a number of rectangles $N$, i.e. $A_i = b \times h_i$ where $h_i = h / N$, then the Parallel-Axis Theorem provides:

$$I_{yy} = \frac{b}{12} \sum h_i^3 + \sum z_i^2 A_i \tag{14}$$

A comparison of the approximation cross section moment of inertia, Equation (12), with the corresponding exact (analytical) cross section moment of inertia, Equation (14), indicates that the numerical integration of the bending moment neglects the contribution of each integration point's local moment of inertia, i.e.

$$\frac{b}{12} \sum h_i^3 = \frac{bh^3}{12} \sum \frac{1}{N^3} = \frac{bh^3}{12} \frac{1}{N^2} \tag{15}$$

For convenience[1], the error can be expressed as the difference between the cross section moment of inertia found from the Parallel-Axis Theorem, $I_{PAT}$ from Equation (14), and the Numerical Approximation, $I_{NA}$ from Equation (12), divided by the analytical cross section moment of inertia, $I_A$ from Equation (7):

[1] The cross section moment of inertia determined from the Parallel-Axis Theorem is identical to the analytical value, but using the numerical form, from the Parallel-Axis Theorem, makes the error illustration (mathematics) easier.

$$Error = \frac{I_{PAT} - I_{NA}}{I_A} = \frac{bh^3/12N^2}{bh^3/12} = \frac{1}{N^2} \tag{16}$$

Equation (16) indicates that the error in the numerical integration is inversely proportional to the number of integration points squared, i.e.

**Table 1** Number of integration points and corresponding numerical integration error.

| $N_i$ | Error $= \dfrac{1}{N^2}$ |
|---|---|
| 2 | 25% |
| 3 | 11.1% |
| 4 | 6.2% |
| 5 | 4.0% |

It is obvious from this table why a minimum of three through-the-thickness integration points are recommended for beam, and shell, elements where bending response is important; even for linear elastic response, as in this simple example. Note: the default LS-DYNA beam integration rule for rectangular, and circular, cross sections is $2 \times 2$ *Guassian Quadrature*, see the parameter QR of the *Beam_Section keyword. The moment of inertial is integrated exactly for this integration rule as 2 point Guassian Quadrature is exact for a quadratic function, e.g. Equations (5) and (7).

Missing from the above table is the case for one integration point, i.e. $N = 1$. For this special case, the error in the numerical integration is essentially infinite, due to the term $z_1 = 0$ in Equation (12).

## EXAMPLES FROM THE LS-DYNA STANDARD CROSS SECTION TYPES

LS-DYNA provides seven standard cross section types, selected via the ICST parameter of the *Integration_Beam keyword, see page 17.4 (Integration) in the v960 User's Manual. These standard cross section types require only a minimum of user input: height, width, and flange thicknesses, from which the predefined integration points are determined. In this section, selected standard cross section types will be used to illustrate how users can specify beam cross section integration rules. Also, based on the above numerical integration error analysis, Equation (16), suggestions on how to improve the standard cross section types are provided.

### W-Section (ICST=1)

The wide flange, W-Section, also know as an I-Section, is the first of the standard cross section types provided by LS-DYNA for beam integration. The default cross section integration uses a total of 9 integration points with 3 points in each of the flanges (horizontal "A" parts) and 3 points in the web (vertical "B" part), see **Figure 3**.

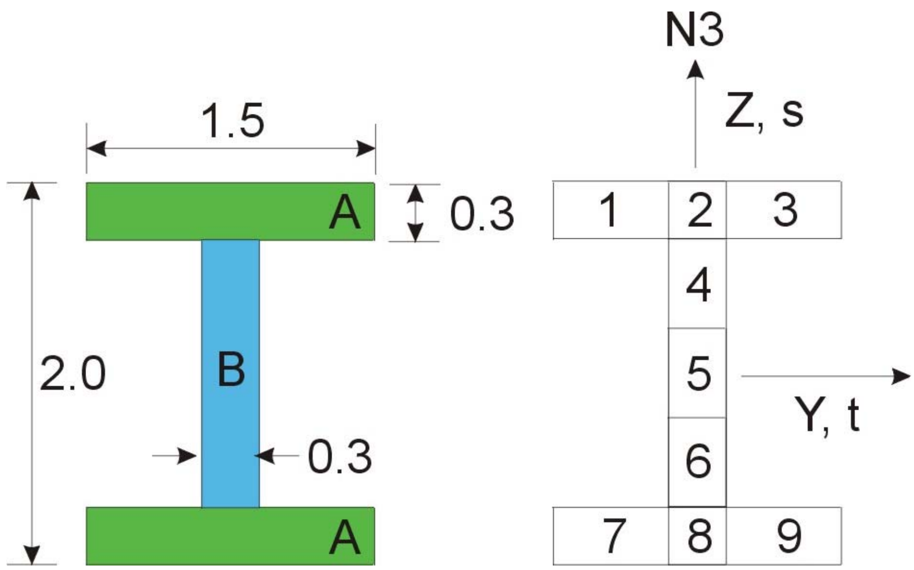


**Figure 3** LS-DYNA provided W-Section (ICST=1) with 9 integration points.

To provide an easy to follow numerical example, specific dimensions are provided in **Figure 3** which correspond to the following LS-DYNA cross section geometry nomenclature, and input parameters:

| Input Parameter | Description | Value |
|---|---|---|
| W | Flange width | 1.5 |
| D | Depth | 2.0 |
| TF | Thickness of the flange ("A") | 0.3 |
| TW | Thickness of the web ("B") | 0.3 |

### *Analytical Integration*

It is useful to calculate the cross section moments of inertia, $I_{yy}$ and $I_{zz}$, for the three regions of the W-section shown on the left in **Figure 3**. The moments of inertia will be calculated about the centrodial axes, which for this doubly symmetric cross section are identical to the mid-height and mid-width axes used by LS-DYNA for all beam integrations.

The calculation of the cross section moments of inertia, and area, is most easily performed with the aid of a spreadsheet application, or similar engineering calculation tool. The following table illustrates the use of a spreadsheet for these calculations:

**Table 2** Analytical integration of W-Section using three parts (A, B, A).

| | | | | Local | | | | Global | |
|---|---|---|---|---|---|---|---|---|---|
| Part | b | h | Area | Iyy | Izz | y^2 | z^2 | Iyy | Izz |
| A | 1.5 | 0.3 | 0.45 | 0.0034 | 0.0844 | 0 | 0.7225 | 0.3285 | 0.0844 |
| B | 0.3 | 1.4 | 0.42 | 0.0686 | 0.0032 | 0 | 0 | 0.0686 | 0.0032 |
| A | 1.5 | 0.3 | 0.45 | 0.0034 | 0.0844 | 0 | 0.7225 | 0.3285 | 0.0844 |
| | | | 1.32 | | | | | 0.7256 | 0.1719 |

The area is the product of the base times the height (bh), where these dimensions are defined for the direct calculation of the local $I_{yy}$ moment of inertia, and are interchanged for the calculation of the local $I_{zz}$ moment of inertia, i.e.

$$I_{yy} = \frac{bh^3}{12} \quad \text{and} \quad I_{zz} = \frac{b^3 h}{12} \tag{17}$$

Next the Parallel-Axis Theorem is used to calculate the global cross section moments of inertia, by adding to the local cross section moments of inertia the product of the area and the squared distance between the local section and the global axes, e.g.

$$I_{yy} = \frac{bh^3}{12} + bhz^2 \tag{18}$$

The three part (A, B, A) results are summed on the last line of the above **Table 2**, and are summarized as

$$\begin{aligned} A &= 1.32 \\ I_{yy} &= 0.7256 \\ I_{zz} &= 0.1719 \end{aligned} \tag{19}$$

### *Analytical 9 Point Integration*

Before proceeding to the numerical integration of the cross section, it is instructive, and useful, to perform the analytical integration in a manner identical to the above 3 part (A, B, A) analytical integration, but using the 9 integration points indicated in **Figure 3**. **Table 3** provides the corresponding spreadsheet entries for the 9 integration points, with the column heading symbols YL and ZL representing the local y and z lengths of each integration point.

**Table 3** Analytical integration of W-Section using 9 points.

| | | | | Local | | | | Global | |
|---|---|---|---|---|---|---|---|---|---|
| IPT | YL | ZL | Area | Iyy | Izz | y^2 | z^2 | Iyy | Izz |
| 1 | 0.6 | 0.3 | 0.18 | 0.0014 | 0.0054 | 0.2025 | 0.7225 | 0.1314 | 0.0419 |
| 2 | 0.3 | 0.3 | 0.09 | 0.0007 | 0.0007 | 0 | 0.7225 | 0.0657 | 0.0007 |
| 3 | 0.6 | 0.3 | 0.18 | 0.0014 | 0.0054 | 0.2025 | 0.7225 | 0.1314 | 0.0419 |
| 4 | 0.3 | 0.4667 | 0.14 | 0.0025 | 0.0011 | 0 | 0.2178 | 0.0330 | 0.0011 |
| 5 | 0.3 | 0.4667 | 0.14 | 0.0025 | 0.0011 | 0 | 0 | 0.0025 | 0.0011 |
| 6 | 0.3 | 0.4667 | 0.14 | 0.0025 | 0.0011 | 0 | 0.2178 | 0.0330 | 0.0011 |
| 7 | 0.6 | 0.3 | 0.18 | 0.0014 | 0.0054 | 0.2025 | 0.7225 | 0.1314 | 0.0419 |
| 8 | 0.3 | 0.3 | 0.09 | 0.0007 | 0.0007 | 0 | 0.7225 | 0.0657 | 0.0007 |
| 9 | 0.6 | 0.3 | 0.18 | 0.0014 | 0.0054 | 0.2025 | 0.7225 | 0.1314 | 0.0419 |
| | | | 1.32 | | | | | 0.7256 | 0.1719 |

Not surprisingly, the total area, and two moments of inertia, are identical with those obtained above using the same analytical integration, but with only 3 parts, i.e. the Parallel-Axis Theorem works for any number or size of rectangular regions.

The utility of the above exercise, when developing user supplied cross section integration rules, is to provides a check on the description of the cross section parameters, YL, ZL, and the corresponding distance parameters, $y^2$ and $z^2$, for all of the integration points, by comparison with the area and moments of inertia from the analytical results obtained using a few large rectangular parts, e.g. A, B, A.

### *Numerical 9 Point Integration*

**Table 4** provides the corresponding spreadsheet entries for the 9 point numerical integration of the W-Section shown previously in **Figure 3**. The columns are similar to those presented in **Table 3** with the important exception that the local moment of inertias are not calculated. Recall from Equation (12), the numerical approximation of the cross section moment of inertia integration, ignores the local moments of inertia, and uses only the local area multiplied by the distance squared to the reference axis, e.g.

$$I_{tt} = \sum z_i^2 A_i \tag{20}$$

where now the subscript "tt" is introduced to indicate this is the numerical approximation of the moment of inertia with respect to the local t-axis; refer back to **Figure 3**, and note that the local s and t axes are oriented with the s-axis pointing towards the 'orientation node,' indicated as "N3" at the top of the right side figure.

**Table 4** Numerical integration of W-Section using 9 points.

| IPT | YL | ZL | Area | Yi | Zi | Itt=Area*Zi^2 | Iss=Area*Yi^2 | Si | Ti | WF |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.6 | 0.3 | 0.18 | -0.45 | 0.85 | 0.1301 | 0.0365 | 0.85 | -0.6 | 0.1364 |
| 2 | 0.3 | 0.3 | 0.09 | 0 | 0.85 | 0.0650 | 0 | 0.85 | 0 | 0.0682 |
| 3 | 0.6 | 0.3 | 0.18 | 0.45 | 0.85 | 0.1301 | 0.0365 | 0.85 | 0.6 | 0.1364 |
| 4 | 0.3 | 0.4667 | 0.14 | 0 | 0.4667 | 0.0305 | 0 | 0.4667 | 0 | 0.1061 |
| 5 | 0.3 | 0.4667 | 0.14 | 0 | 0 | 0 | 0 | 0 | 0 | 0.1061 |
| 6 | 0.3 | 0.4667 | 0.14 | 0 | -0.4667 | 0.0305 | 0 | -0.4667 | 0 | 0.1061 |
| 7 | 0.6 | 0.3 | 0.18 | -0.45 | -0.85 | 0.1301 | 0.0365 | -0.85 | -0.6 | 0.1364 |
| 8 | 0.3 | 0.3 | 0.09 | 0 | -0.85 | 0.0650 | 0 | -0.85 | 0 | 0.0682 |
| 9 | 0.6 | 0.3 | 0.18 | 0.45 | -0.85 | 0.1301 | 0.0365 | -0.85 | 0.6 | 0.1364 |
| | | | 1.32 | | | 0.7112 | 0.1458 | | | 1 |
| | | | | | Analytical | 0.7256 | 0.1719 | | | |
| | | | | | Error | -2.0% | -15.2% | | | |

**Table 4** shows that the total area is calculated correctly, but the approximate moments of inertia under estimate[2] the corresponding analytical values by 2% for $I_{tt}$ and 15.2% for $I_{ss}$. These numerical integration errors are consistent with those derived in **Table 1** for the simple rectangular cross section. That table indicated a 4% error, when 5 integration points are used, as is the case in the present example along the z-axis (s-axis), and corresponding $I_{tt}$ error of 2%. For the $I_{ss}$ integration, the 15.2% error results from using only 3 integration points along the y-axis (t-axis), and the corresponding 3 point error from **Table 1** is 11%.

This suggests that if the -15.2% integration error for the $I_{ss}$ moment of inertia, and hence the elastic stiffness of the corresponding beam element, is excessive for a particular application, the number of integration points along the y-axis (t-axis) should be increased. This is demonstrated in the next subsection.

**Table 4** also provides the corresponding LS-DYNA inputs for this cross section integration rule in the last three columns. The LS-DYNA inputs consist of non-dimensionalized coordinates along the s and t axes, and the relative area of each integration point, i.e.

$$
\begin{aligned}
S_i &= \frac{z_i}{D/2} = \frac{2z_i}{D} \quad (-1 < S_i < 1) \\
T_i &= \frac{y_i}{W/2} = \frac{2y_i}{W} \quad (-1 < T_i < 1) \\
WF_i &= \frac{A_i}{A} = \frac{(YL)(ZL)}{A} \quad \left(\sum_i WF_i = 1\right)
\end{aligned}
\tag{21}
$$

The other LS-DYNA input, required for user defined integration rules, is the relative area of the entire cross section, e.g.

$$RA = \frac{A}{WD} \tag{22}$$

[2] Negative values of the errors, in the tables, indicate the numerical prediction *under* estimates the analytical value.

These non-dimensionalized cross section parameters are used to integrate the stress over the cross section and determine the axial force and moments as described in the appendix Non-dimensionalized Cross Section Integration Formula.

### *Numerical 11 Point Integration*

The LS-DYNA default 9 point integration of the W-Section can be made more accurate, with respect the to $I_{ss}$ moment of inertia, with the addition of two more integration points; one for each of the horizontal flanges, see **Figure 4**. Using the same overall W-Section dimensions, see **Figure 3**, **Table 5** provides the corresponding spreadsheet entries for the depicted 11 point integration.

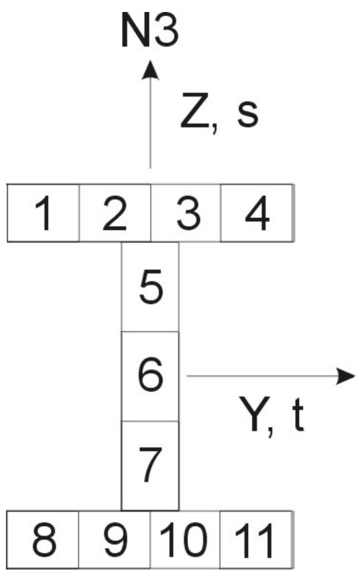


**Figure 4** Example of an 11 point integration of a W-Section.

**Table 5** Numerical integration of W-Section using 11 points.

| IPT | YL | ZL | Area | Yi | Zi | Itt=Area*Zi^2 | Iss=Area*Yi^2 | Si | Ti | WF |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.375 | 0.3 | 0.1125 | -0.5625 | 0.85 | 0.0813 | 0.0356 | 0.85 | -0.75 | 0.0852 |
| 2 | 0.375 | 0.3 | 0.1125 | -0.1875 | 0.85 | 0.0813 | 0.0040 | 0.85 | -0.25 | 0.0852 |
| 3 | 0.375 | 0.3 | 0.1125 | 0.1875 | 0.85 | 0.0813 | 0.0040 | 0.85 | 0.25 | 0.0852 |
| 4 | 0.375 | 0.3 | 0.1125 | 0.5625 | 0.85 | 0.0813 | 0.0356 | 0.85 | 0.75 | 0.0852 |
| 5 | 0.3 | 0.4667 | 0.14 | 0 | 0.4667 | 0.0305 | 0 | 0.4667 | 0 | 0.1061 |
| 6 | 0.3 | 0.4667 | 0.14 | 0 | 0 | 0 | 0 | 0 | 0 | 0.1061 |
| 7 | 0.3 | 0.4667 | 0.14 | 0 | -0.4667 | 0.0305 | 0 | -0.4667 | 0 | 0.1061 |
| 8 | 0.375 | 0.3 | 0.1125 | -0.5625 | -0.85 | 0.0813 | 0.0356 | -0.85 | -0.75 | 0.0852 |
| 9 | 0.375 | 0.3 | 0.1125 | -0.1875 | -0.85 | 0.0813 | 0.0040 | -0.85 | -0.25 | 0.0852 |
| 10 | 0.375 | 0.3 | 0.1125 | 0.1875 | -0.85 | 0.0813 | 0.0040 | -0.85 | 0.25 | 0.0852 |
| 11 | 0.375 | 0.3 | 0.1125 | 0.5625 | -0.85 | 0.0813 | 0.0356 | -0.85 | 0.75 | 0.0852 |
| | | | 1.32 | | | 0.7112 | 0.1582 | | | 1 |
| | | | | | | 0.7256 | 0.1719 | | | |
| | | | | | % Error | -2.0% | -8.0% | | | |

**Table 5** indicates the numerical integration error in the $I_{ss}$ moment of inertia is reduced to -8% from the previous -15.2% when 9 integration points were used; e.g. 3 points along the y-axis (t-axis). Often in engineering applications, errors on the order of 10% are acceptable, but the error could be further reduced by adding another two integration points along the y-axis (t-axis).

The integration template shown in **Figure 4** also illustrates that the numerical integration points do not need to form continuous, or consistent, interfaces of the rectangular regions, i.e. there is no need to partition the horizontal flange integration regions to account for the width of the vertical web, as was done in **Figure 3**. Further, the numerical integration regions need not be contiguous, i.e. unconnected or isolated integration regions are permitted.

### Angle Section (ICST=3)

The Angle Section, is the third of the standard cross section types provided by LS-DYNA for beam integration. The default cross section integration uses a total of 5 integration points, see **Figure 5**. For a 'symmetric' angle section, i.e. where the sides of the angle are (nearly) equal in length, the 5 integration point template does quite well. However, for an 'asymmetric' angle section, the 5 integration point template performs less well.

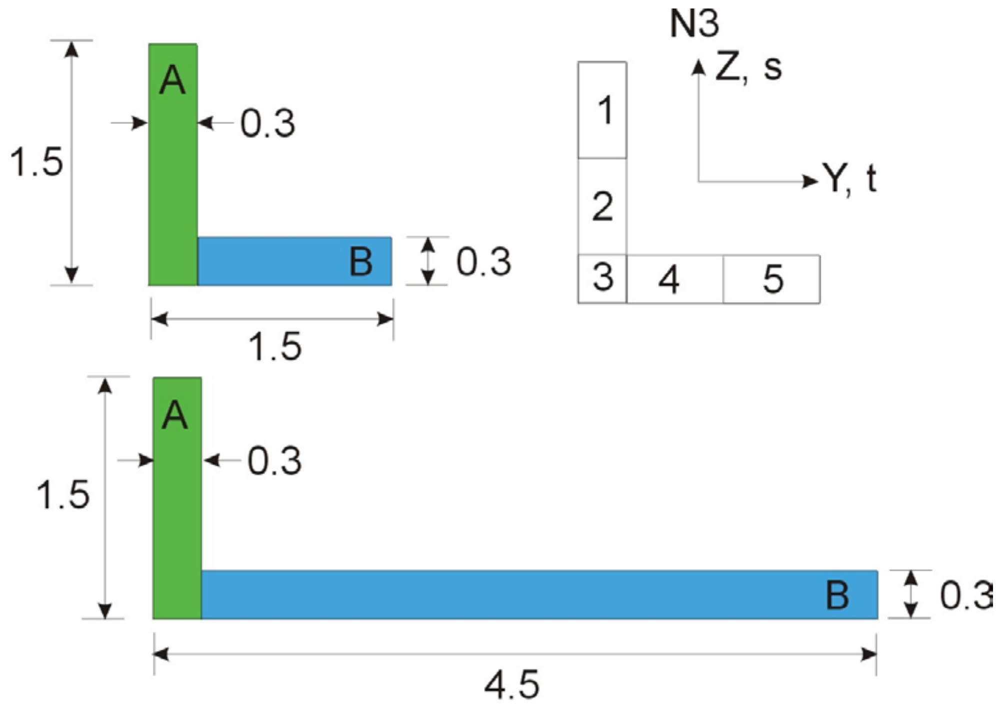


**Figure 5** LS-DYNA provided Angle Section (ICST=3) with 5 integration points.

To provide an easy to follow numerical example, specific dimensions are provided in **Figure 5** which correspond to the following LS-DYNA cross section geometry nomenclature, and input parameters:

| Input Parameter | Description | Symmetric | Asymmetric |
|---|---|---|---|
| W | Flange width | 1.5 | 4.5 |
| D | Depth | 1.5 | 1.5 |
| TF | Thickness of the flange ("A") | 0.3 | 0.3 |
| TW | Thickness of the web ("B") | 0.3 | 0.3 |

### *Symmetric Angle Section*

For the symmetric Angle Section, use of the Parallel-Axis Theorem, on the two parts (A, B), provides the analytical integration of the cross section moments of inertia as shown in **Table 6**.

**Table 6** Analytical integration of the symmetric Angle Section.

| | | | | Local | | | | Global | |
|---|---|---|---|---|---|---|---|---|---|
| Part | b | h | Area | Iyy | Izz | y^2 | z^2 | Itt | Iss |
| A | 0.3 | 1.5 | 0.45 | 0.084375 | 0.003375 | 0.36 | 0 | 0.0844 | 0.1654 |
| B | 1.2 | 0.3 | 0.36 | 0.0027 | 0.0432 | 0.023 | 0.36 | 0.1323 | 0.0513 |
| | | | 0.81 | | | | | 0.2167 | 0.2167 |

It should be noted that the LS-DYNA local mid-distance coordinate system, s and t, is *not* a centrodial coordinate system for any nominal configuration of an Angle Section.

The 5 point numerical integration of the symmetric Angle Section is summarized in **Table 7**. For this geometry the 5 point integration does a very good job of estimating the analytical moments of inertia with only a 6.5% under estimation. Since only 3 points are used along each axis, the expected error would be about 11%, as reported in **Table 1**. By changing the cross section, to an asymmetric Angle Section, the error, in one of the cross section moments of inertia, will be shown to increase.

**Table 7** Numerical integration of symmetric Angle Section using 5 points.

| IPT | YL | ZL | Area | Yi | Zi | Itt=Area*Zi^2 | Iss=Area*Yi^2 | Si | Ti | WF |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.3 | 0.6 | 0.18 | -0.6 | 0.45 | 0.0365 | 0.0648 | 0.6 | -0.8 | 0.2222 |
| 2 | 0.3 | 0.6 | 0.18 | -0.6 | -0.15 | 0.0041 | 0.0648 | -0.2 | -0.8 | 0.2222 |
| 3 | 0.3 | 0.3 | 0.09 | -0.6 | -0.6 | 0.0324 | 0.0324 | -0.8 | -0.8 | 0.1111 |
| 4 | 0.6 | 0.3 | 0.18 | -0.15 | -0.6 | 0.0648 | 0.0041 | -0.8 | -0.2 | 0.2222 |
| 5 | 0.6 | 0.3 | 0.18 | 0.45 | -0.6 | 0.0648 | 0.0365 | -0.8 | 0.6 | 0.2222 |
| | | | 0.81 | | | 0.2025 | 0.2025 | | | 1 |
| | | | | | Analytical | 0.2167 | 0.2167 | | | |
| | | | | | Error | -6.5% | -6.5% | | | |

***Asymmetric Angle Section***

For the asymmetric Angle Section, shown previously in **Figure 5**, the Parallel-Axis Theorem provides the analytical integration of the cross section moments of inertia as shown in **Table 8**.

**Table 8** Analytical integration of the asymmetric Angle Section.

| | | | | Local | | | | Global | |
|---|---|---|---|---|---|---|---|---|---|
| Part | b | h | Area | Iyy | Izz | y^2 | z^2 | Itt | Iss |
| A | 0.3 | 1.5 | 0.45 | 0.0844 | 0.0034 | 4.41 | 0 | 0.0844 | 1.9879 |
| B | 4.2 | 0.3 | 1.26 | 0.0095 | 1.8522 | 0.0225 | 0.36 | 0.4631 | 1.8806 |
| | | | 1.71 | | | | | 0.5474 | 3.8684 |

The 5 point numerical integration of the asymmetric Angle Section is summarized in **Table 9**. For this geometry, the 5 point integration does a very good job of estimating the analytical $I_{tt}$ moment of inertia, i.e. about the t-axis (y-axis) with 3.8% under estimation. However, the estimate for the $I_{ss}$ moment of inertia, i.e. about the s-axis (z-axis), has an under estimation error of 12.1%

**Table 9** Numerical integration of asymmetric Angle Section using 5 points.

| IPT | YL | ZL | Area | Yi | Zi | Itt=Area*Zi^2 | Iss=Area*Yi^2 | Si | Ti | WF |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.3 | 0.6 | 0.18 | -2.1 | 0.45 | 0.0365 | 0.7938 | 0.6 | -0.9333 | 0.1053 |
| 2 | 0.3 | 0.6 | 0.18 | -2.1 | -0.15 | 0.0041 | 0.7938 | -0.2 | -0.9333 | 0.1053 |
| 3 | 0.3 | 0.3 | 0.09 | -2.1 | -0.6 | 0.0324 | 0.3969 | -0.8 | -0.9333 | 0.0526 |
| 4 | 2.1 | 0.3 | 0.63 | -0.9 | -0.6 | 0.2268 | 0.5103 | -0.8 | -0.4 | 0.3684 |
| 5 | 2.1 | 0.3 | 0.63 | 1.2 | -0.6 | 0.2268 | 0.9072 | -0.8 | 0.5333 | 0.3684 |
| | | | 1.71 | | | 0.5265 | 3.4020 | | | 1 |
| | | | | | Analytical | 0.5474 | 3.8684 | | | |
| | | | | | Error | -3.8% | -12.1% | | | |

The asymmetric Angle Section error in the $I_{ss}$ moment of inertia is largely due to the poor location of the integration points, especially point 3, along the long horizontal side of the angle. The 5 point integration template, shown previously in **Figure 5**, is not optimal for an asymmetric Angle Section. A better choice of the 5 point integration template for the asymmetric cross section is shown in **Figure 6**, where the 3 integration points along the horizontal leg are evenly distributed along this length.

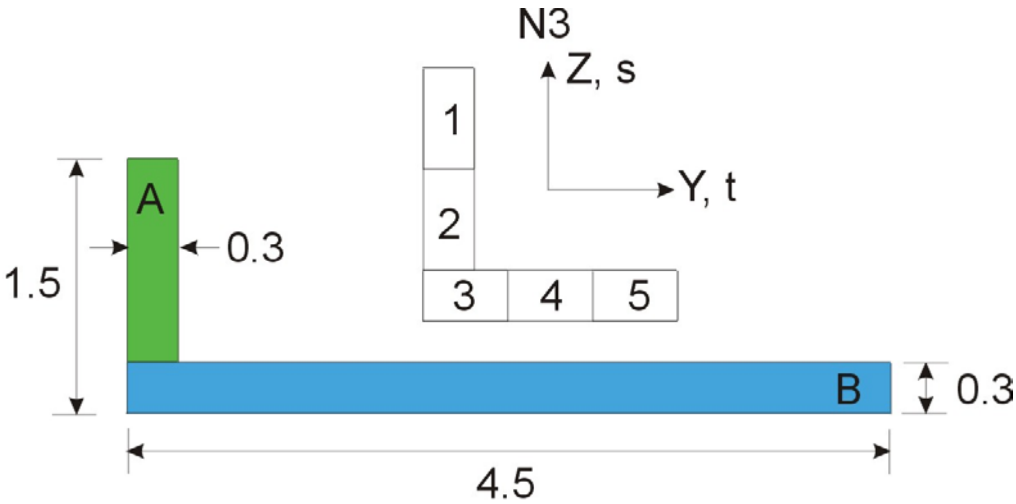


**Figure 6** Alternative 5 point integration for asymmetric Angle Section.

The alternative 5 point numerical integration of the asymmetric Angle Section is summarized in **Table 10**. The alternative numerical integration reduces the error in the $I_{ss}$ moment of inertia to an under prediction of 6.6%, almost half the error calculated when the LS-DYNA provided integration template is used.

**Table 10** Alternative 5 point numerical integration of asymmetric Angle Section.

| IPT | YL | ZL | Area | Yi | Zi | Itt=Area*Zi^2 | Iss=Area*Yi^2 | Si | Ti | WF |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.3 | 0.6 | 0.18 | -2.1 | 0.45 | 0.0365 | 0.7938 | 0.6 | -0.9333 | 0.1053 |
| 2 | 0.3 | 0.6 | 0.18 | -2.1 | -0.15 | 0.0041 | 0.7938 | -0.2 | -0.9333 | 0.1053 |
| 3 | 1.5 | 0.3 | 0.45 | -1.5 | -0.6 | 0.1620 | 1.0125 | -0.8 | -0.6667 | 0.2632 |
| 4 | 1.5 | 0.3 | 0.45 | 0 | -0.6 | 0.1620 | 0.0000 | -0.8 | 0 | 0.2632 |
| 5 | 1.5 | 0.3 | 0.45 | 1.5 | -0.6 | 0.1620 | 1.0125 | -0.8 | 0.6667 | 0.2632 |
| | | | 1.71 | | | 0.5265 | 3.6126 | | | 1 |
| | | | | | Analytical | 0.5474 | 3.8684 | | | |
| | | | | | Error | -3.8% | -6.6% | | | |

## SUMMARY

The LS-DYNA provided beam cross section integration templates are a great convenience for the user. However, the user should always check that the provided integration template is adequate for the present application.

The detailed examples, and cross section integration explanations, provided in this article, should allow users to easily create their own user supplied beam integration rules, and control the level of error caused by the numerical approximation of the cross section integration.

In addition to the spreadsheet types of checks provided in this article, users are urged to also construct simple elastic beam models, e.g. cantilever beam under an end load, to assess the accuracy of their beam cross section integrations.

## APPENDIX NON-DIMENSIONALIZED CROSS SECTION INTEGRATION FORMULA

The integration of the stress over the cross section to determine the axial force and moments, uses non-dimensionalized user supplied integration parameters. The user supplied parameters are the non-dimensional coordinates of the integrations points, $s_i$ and $t_i$, the relative area of the integration point $WF_i$, and the relative area of the entire cross section $RA$; refer back to Equations (21) and (22) for definitions. In addition, the overall cross section dimensions, TS and TT, also need to be provided via the *Section_Beam keyword; in the W-Section example provided, see **Figure 3**, TT=W and TS=D.

Using this non-dimensional cross section notation, the area of each integration point is given by

$$A_i = (WF_i)(RA)(TS)(TT) \tag{23}$$

Then the numerical integration for the axial force, Equation (9), and moment, Equation (10), are given by:

$$F = \int \sigma dA \approx \sum \sigma_i A_i = (RA)(TS)(TT)\sum \sigma_i (WF_i) \tag{24}$$

and

$$\begin{aligned} M_y &= \int \sigma z dA \approx \sum \sigma_i z_i A_i = (RA)(TS)(TT)\sum \sigma_i z_i (WF_i) \\ &= (RA)(TS)(TT)\sum \sigma_i \left(\frac{TS}{2} s_i\right)(WF_i) \\ &= \frac{1}{2}(RA)(TS)^2 (TT)\sum \sigma_i s_i (WF_i) = M_t \end{aligned} \tag{25}$$

where the moment about the s-axis is obviously

$$M_s = \frac{1}{2}(RA)(TS)(TT)^2 \sum \sigma_i t_i (WF_i) \tag{26}$$

## APPENDIX SIMPLE FORMULA FOR INTEGRATION POINTS ON A RECTANGULAR STRIP

Consider a rectangular strip of width W. If the y-coordinate origin is located at the mid-width of the strip, i.e. W/2, then for any number integration points N, aligned in the y-direction, the y-coordinate is given by:

$$y_i = \left[ i - \frac{N+1}{2} \right]\left( \frac{W}{N} \right)$$