# Analyzing 'Noisy' Structural Problems with LS-OPT: Probabilistic and Deterministic Fundamentals

Willem Roux and Nielen Stander

*Livermore Software Technology Corporation, Livermore CA*

Keywords: Probabilistic Analysis, Metamodeling, Structural Reliability Analysis

***Abstract***

System identification of 'noisy' structural design optimization problems: the sources of uncertainty, the competing roles of bias and variance, and the interaction of uncertainty and deterministic effects. Two test problems are used to clarify the effect of different approaches.

## 1. Introduction

No two structural events will be exactly similar; nor will a structural event occur exactly as designed or analyzed. Adverse combinations of design and loading variation may lead to undesirable behavior or failure; therefore, if significant variation is expected, a probabilistic evaluation is required.

The responses may be known only in probabilistic terms in certain classes of structural problems, for example: vehicle crash. Computational analysis may be the most time and cost efficient solution due to large number of data associated with different designs that must be collected.

Differences in structural performances can be attributed to deterministic and random effects. The redesign of the structure requires understanding the link between cause and effect, while the variance of the response about its nominal is required for judgments such as reliability. Distinguishing between deterministic and random effects is however challenging.

Having repeatability of results is desirable. Reducing the variation in the responses requires that the sources of the variation be well understood. A number of sources of uncertainty are applicable for reliability analysis [1]; all of these must be considered for applicable results. The resulting reliability evaluation procedure can be used to redesign the structure, or inside a reliability design optimization scheme [2] to automatically find an improved design.

This study accordingly describes the sources and quantification of the variation of an explicit FEA analysis.

## 2. Probabilistic Background

### 2.1 Response Variation

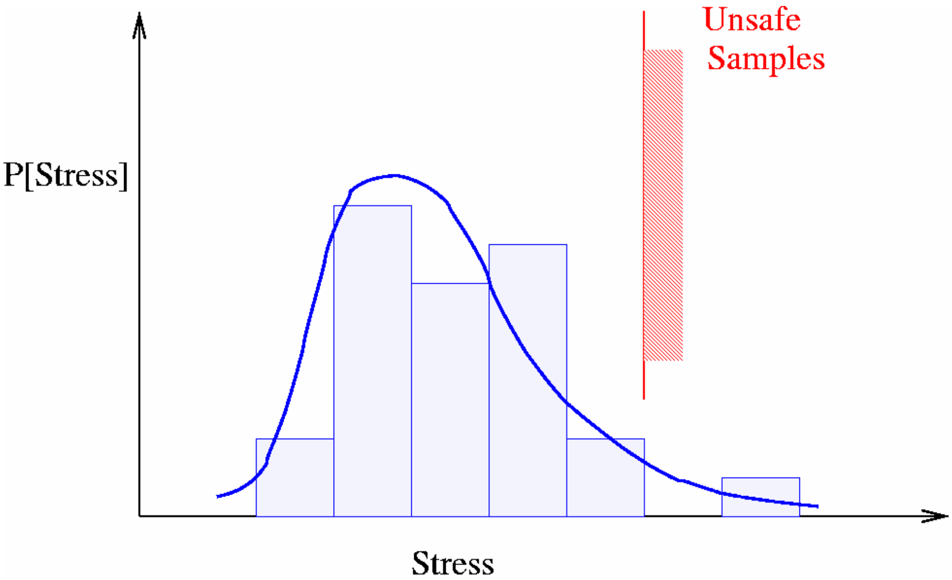


**Figure 1: Response Variation**

The variation of the response can be decomposed as:

- ***Deterministic variation.*** An expected, predictable, and repeatable variation in a response associated with a variation in a parameter.
- ***Random variation.*** Variation that cannot be associated with a change in the system parameters. The random variation can be further decomposed as:
    - Regular random variation. Not known to be associated with the physics of the problem.
    - Chaotic random variation. Noise caused to grow because of bifurcation (eigenvalue) behavior in the structure.

The response variation is quantified using a probabilistic analysis. From the probabilistic analysis we want to infer the following from the responses:

- Distribution of the response values.
- Probability of failure.
- Properties of the designs associated with failure.
    - Variable screening - identify important noise factors.
    - Dispersion factors - factors whose settings may increase variability of the responses.
- Efficient redesign strategies.
- Understand the source of noise in structure.

### 2.2 Sources of Variation

The variation of the responses is caused by:

- Variation in the structure; for example: variation in yield stress.
- Variation in the environment; for example: variation in a load.
- Variation in the problem modeling and analysis; for example, mesh density.
- Variation in the analysis; for example, a different buckling mode being activated.
- Pure variation; for example, machine precision.

#### 2.2.1 Design Parameter Variation

The variation in a response due to a variation in a variable is usually known as a deterministic relationship computed using FEA. Both the variation of the structure and the variation of the environment can be described using design parameters.

Considering the sources of uncertainty in the system parameters, we decompose the variables into two classes (using the Taguchi naming convention):

- **Control variables**: Variables that can be controlled in the design, analysis, and production level; for example: a shell thickness. It can therefore be assigned a nominal value and will have a variation around this nominal value. The nominal value can be adjusted during the design phase in order to have a more suitable design.
- **Noise variables:** Variables that are difficult or impossible to control at the design and production level, but can be controlled at the analysis level; for example: loads and material variation. A noise variable will have the nominal value as specified by the distribution; that is, it will follow the distribution exactly.

The relationship between the control process variables and the response variance can be used to adjust the control variables in order to have an optimum process. The variance of the control and noise variables can be used to predict the variance of the system, which may then be used for redesign. Knowledge of the interaction between the control and noise variables can be valuable; for example, information such that the dispersion effect of the material variation (a noise variable), may be less at a high process temperature (a control variable) can be used to selected control variables for a more robust design.

#### 2.2.2 Modeling Variation

Differences in modeling will give different results as well as introduce noise into the results. Amongst others, the following factors:

- Mesh density.
- Choice between FEA, Element Free Galerkin, and SPH methods.
- Resolution of data gathered – time step and filtering selection.
- Selection of node/element to monitor.

The physics of the problem must of course be modeled correctly for the above to be relevant.

### 2.2.3 Analysis Variation

Slightly different initial conditions, especially when driven by eigenvalues, can lead to noticeable differences in responses.

- Physical:
  - Bifurcation events can be sensitive to initial values; for example, buckling initiation.
  - Changes in the design variables may cause different components to come into contact, or change the order of impact.
- Algorithmic:
  - Contact algorithms. The discretization of a smooth structure into piecewise linear finite elements may lead to different orders of events; for example, a node impacting the edge of a given element may impact the edge of its neighboring element in a similar simulation.

### 2.2.4 Pure Variation

This is a change in results associated with trivial or unrelated change in inputs; for example: running on a different machine. Machine precision and compiler differences are the important sources here.

These small changes can lead to a larger change of the response values if it triggers a (different) bifurcation.

## 2.3 Reliability Computations

A number of methods exist for reliability computations. We consider the following set:

- Monte Carlo Simulation
- Using the standard deviation e.g. six sigma.
- Most probable point (MPP) based methods; for example FORM.

All of the above methods can be used together with metamodels. The error of the fit of the metamodel, amongst others, contributes to the total error of computing reliability [3].

Monte Carlo simulation is very expensive for the computation of small probabilities. A suggestion for the minimum sampling size provided by reference [2] is: $N = 10 / P[G(x) \leq 0]$ with *P* the probability being estimated; therefore indicating about 100 FE evaluations to evaluation an event with a 10% probability and 1000 FE evaluations for an event with a 1% probability. Probability computations are therefore usually used together with metamodels for which millions of functions evaluations are feasible. If only the mean value and the standard deviation of the response is desired then the method is more attractive, and may be the best method in some cases. The error of estimating the mean is related to $\sqrt{N}$ with N the number of design variables.

Another way of using the Monte Carlo results is computing the mean value and standard deviation (sigma) of the response and using the number of sigma's away from failure as an indication of reliability.

The probability of failure associated with a certain sigma value can then be approximated by assuming a normally distribution response. For these computations, we compute β, the reliability index, as:

$$\beta = \frac{E[G(X)]}{D[G(X)]}$$

with E and D the expected value and standard deviation operators respectively. The probability of failure is then computed as:

$$P_f = \Phi(-\beta)$$

with $\Phi(x)$ the cumulative normal distribution function.

Accurate computation of low values of failure (high reliability) is usually done using method computing the Most Probable Point of failure (MPP). The advantages of these methods are: (i) the MPP gives an indication of the design most likely to fail and (ii) highly accurate reliability methods utilizing an approximation around the MPP are possible. In this study we use FORM together with the Hasofer-Lind transformation (see [1] for more details).

What constitutes engineering accuracy at the low probabilities is an open question. A definition such as six-sigma may be best way of specifying the engineering requirement; a precise numerical value may be not be meaningful. The accuracy of the probability of failure computation should however be such that different designs can be compared with each other.

Much more accurate and sophisticated methods of computing reliability are available [1, 3, 4], but are outside the scope of this study.

### 2.4 Competing roles of variance and bias

In an investigation the important design variables are varied while other sources are kept at a constant value in order to minimize their influence. In practice the other sources will have an influence. Distinguishing whether a difference in a response value is due to a deterministic effect or other variation is difficult, because both always have a joint effect in the computer experiments being considered.

In general [5] the relationship between the responses **y** and the variables **x** is:

$$\mathbf{y} = f(\mathbf{x}) + \delta(\mathbf{x}) + \mathbf{e}$$

with f(**x**) the metamodel; $\delta(\mathbf{x}) = \eta(\mathbf{x}) - f(\mathbf{x})$, the bias, is the difference between the chosen metamodel and the true functional response $\eta(\mathbf{x})$; and $\mathbf{e}$ is the random deviation.

The bias (fitting error) and variance component both contribute to the residuals. If we compute the variance of the random deviation using the residuals then the bias component is included in our estimate of the variance.

## 3. Examples

### 3.1 Basic Problem – Two bar truss

We use the two bar truss problem as shown in the figure to demonstrate some concepts.

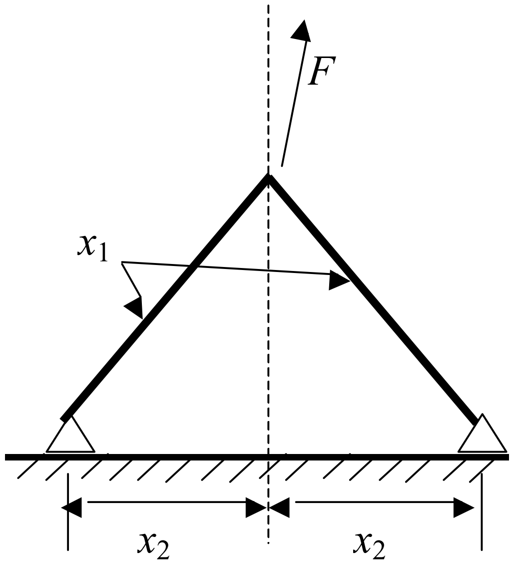


**Figure 2: Two-bar truss example**

There are two design variables: $x_1$ the cross-sectional area of the bars, and $x_2$ half of the distance (*m*) between the supported nodes. The lower bounds on the variables are 0.2$cm^2$ and 0.1*m*, respectively. The upper bounds on the variables are 4.0$cm^2$ and 1.6*m*, respectively.

The stress is constrained as follows:

$$\sigma(x) = 0.124\sqrt{1+x_2^2}\left(\frac{8}{x_1}+\frac{1}{x_1 x_2}\right) \leq 1$$

The probability of violating the stress constraint is computed. The design variables are normally distributed with a standard deviation of 0.05. In addition to the variation due to the design variables, a normally distributed random component with a standard deviation 0.025 is added to the stress results.

We compare the results from a number of strategies of computing reliability at different design points with the results as shown in the table.

| **Method** | **Design** | | | | |
|---|---|---|---|---|---|
| | [1.40, 0.5] | [1.45, 0.5] | [1.50, 0.5] | [1.55, 0.5] | [1.60, 0.5] |
| | **Probability of Failure** $P[\sigma(\mathbf{x}) > 1.0]$ **[Probability without random variation compensation]** | | | | |
| Correct | **0.44** | **0.17** | **0.038** | **0.0050** | **0.00036** |
| Monte Carlo (150 experiments) | 0.49 | 0.15 | 0.040 | 0 | 0 |
| Using Reliability Index | 0.48 | 0.17 | 0.033 | 0.0029 | 0.00012 |
| Linear Taylor expansion at design | 0.41 [0.39] | 0.14 [0.09] | 0.028 [0.007] | 0.0028 [0.0001] | 0.00015 [0.00000] |
| Quadratic Taylor expansion at design | 0.44 [0.42] | 0.16 [0.11] | 0.037 [0.013] | 0.00475 [0.0006] | 0.00035 [0.00000] |
| Linear Taylor expansion at MMP | 0.41 [0.39] | 0.15 [0.10] | 0.031 [0.012] | 0.0036 [0.0005] | 0.00025 [0.00000] |
| Quadratic Taylor expansion at MPP | 0.44 [0.42] | 0.17 [0.11] | 0.038 [0.014] | 0.0051 [0.0007] | 0.00038 [0.00001] |
| FORM | 0.43 [0.39] | 0.16 [0.10] | 0.039 [0.0115] | 0.0053 [0.0005] | 0.00046 [0.00001] |

The ‘correct’ probabilities are computed using a Monte Carlo analysis with $10^7$ experiments. For this trivial problem we can evaluate the structure millions of times; in general, using FEA simulations, this is not feasible.

The practical use of Monte Carlo simulation in FEA limits the number of function evaluations to about 150. We use the Monte Carlo simulation to compute probability of failure and as well as the mean value and standard deviation of the response.

Using the mean value and standard deviation of the response computed in the Monte Carlo analysis, we can compute the probability of failure assuming a normally distributed response. These computations start showing errors at probabilities lower than 1% (3 sigma events) for this study. This also has implications for methods such as FORM that depends on the normally distributed assumption.

For the Taylor approximations, we compute the probability using the approximation inside a Monte Carlo simulation using $10^7$ experiments. We compensate for the random variation by adding a normally distributed noise to the approximations when computing the probabilities. In this case the magnitude of the noise is known; in the general case the magnitude of the noise is computed when fitting the response surface – the lack of fit is assumed to be due to the noise.

We create the Taylor approximations around both the current design point and the MPP (Most Probable Point) computed using FORM. The MPP is the closest point on the failure initiation hyperplane; an approximation accurate at the MPP should allow for more accurate reliability computations as confirmed especially for the linear approximation reliability results.

In the FORM we compensate for the noise component by adding it to the variation caused by the variables in the ***u***-space: $\beta_{total} = \beta_{\det er\min istic}\beta_{random} / (\beta^2_{random} + \beta^2_{\det er\min istic})$ .

The results without the noise are given as well in the table. It can be seen that the noise component contributed significantly to the probability of failure this above study. The standard deviation of the response is 0.036 (3.6%) without the noise component and 0.044 (4.4%) with the noise component. This leads to significantly different results at low probabilities of failure.

The quadratic Taylor expansion at the MPP gave the best accuracy of the reliability methods tested. For use in reliability, a metamodel should therefore incorporate curvature and be accurate at the failure hyperplane.

In the above we used Taylor expansions and FORM together with a known variance of the random component. In general the standard deviation of the random component will be computed when fitting the metamodel. To investigate the accuracy of obtaining the random component, we fit a quadratic response surface, using a space filling experimental design, to a subregion size of 0.4 $cm^2$ by 0.2$m$ in the design variables in order to minimize the bias error. Both the standard deviation of the noise and mean value of the response computed from the Monte Carlo analysis are required to compute the reliability; they are given the following table. From the data it seems, that for this problem, more than 50 experiments are required to estimate the second digit of the variance; however, more experience with industrial problems and statistical tests are desirable for recommendations. Note that at least six experiments are required to compute the response surface; the other experiments are required to estimate the variance. The number of experiments required to compute the metamodel increases with the number of design variables, while the number of experiments required to estimate the variance does not increase with the number of design variables.

| Number of experiments | Standard Deviation | Mean |
|---|---|---|
| 10 | 0.0214 | 1.003 |
| 20 | 0.0242 | 0.9866 |
| 50 | 0.0205 | 0.9959 |
| 100 | 0.0256 | 0.9938 |
| 500 | 0.0247 | 0.9941 |

### 3.2 Head Impact Problem

We consider the problem of a Free Motion Headform (FMH) impacting an A-pillar as shown in the figure. The mesh is parameterized using the TrueGrid preprocessor [6]. We consider two variables: the angle of the impact and the rib stiffener height of the pillar padding. The angle of impact is taken to be 15 degrees with a 10 percent standard deviation, normally distributed. The rib height is 12.5mm with a 5 percent standard deviation, also normally distributed. We investigate the variance on the Head Injury Criterion, HIC-d = 166.4 + 0.75466*HIC15.

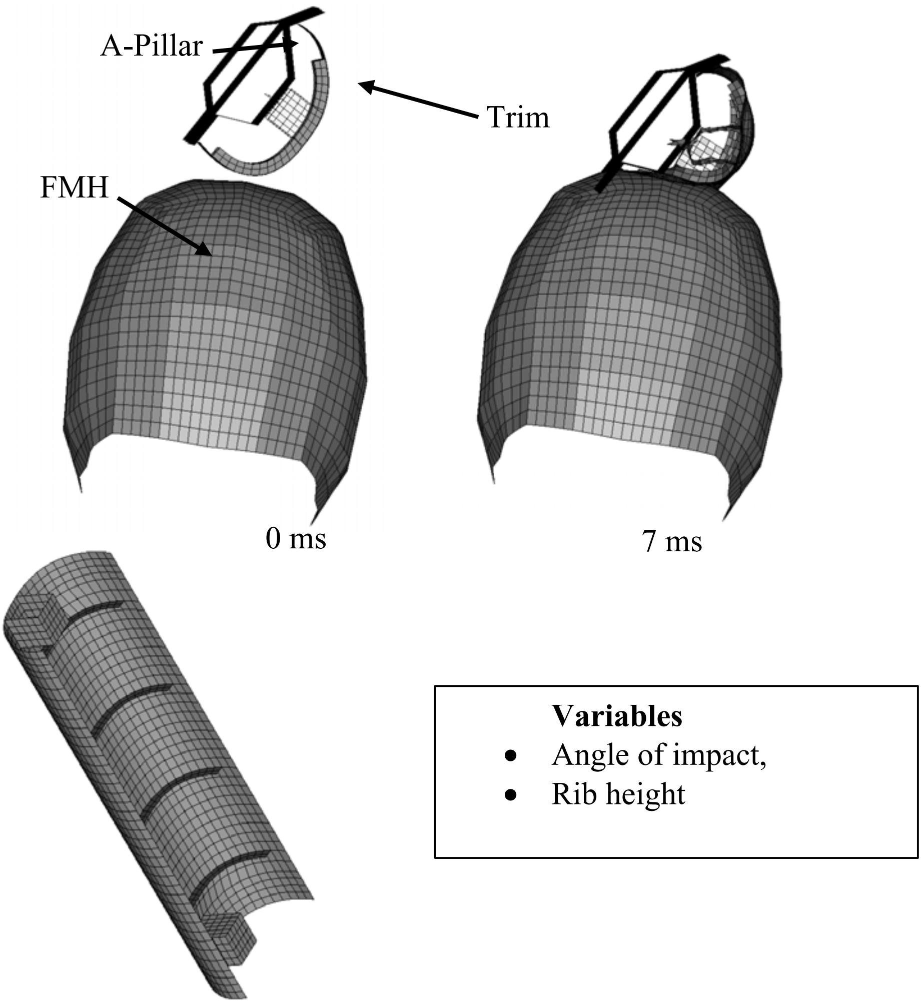


**Figure 3: Head Impact FE Model**

Firstly we investigate the problem using a parametric study in which we vary one of the variables at a time. The results are shown in figure 4 and 5. From the figure it is clear that the problem has some noise and the variables has a mostly linear effect on the HIC response.

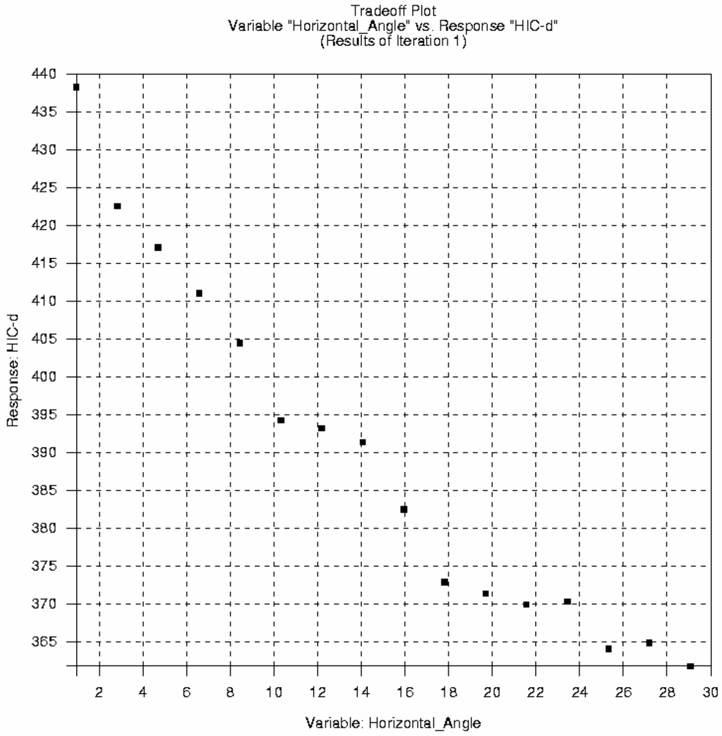


**Figure 4: HIC - Horizontal Angle variation**

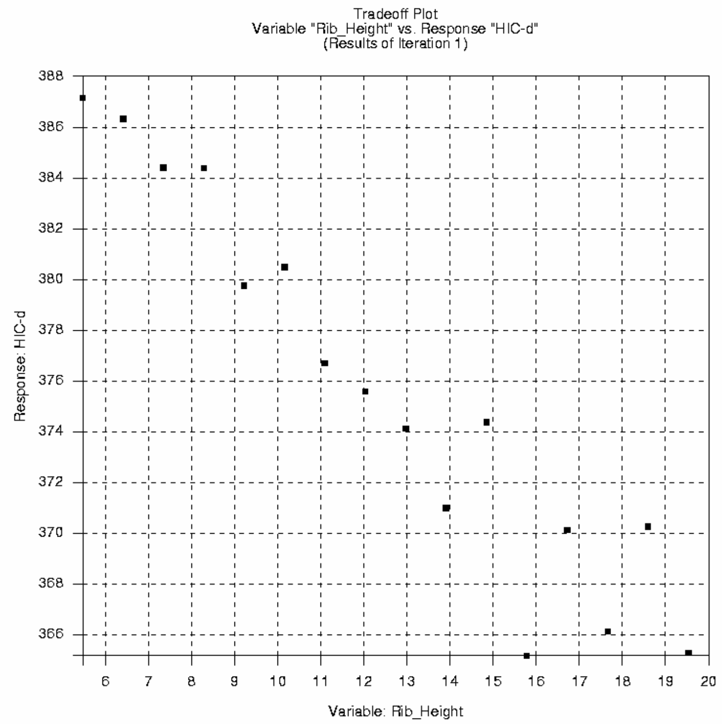


**Figure 5: HIC - Rib Height variation**

We investigate the probability of the HIC-d exceeding certain values using a Monte Carlo analysis and a metamodel.

The Monte Carlo evaluation using 150 points was done with the results as in shown in figures 6 and 7.

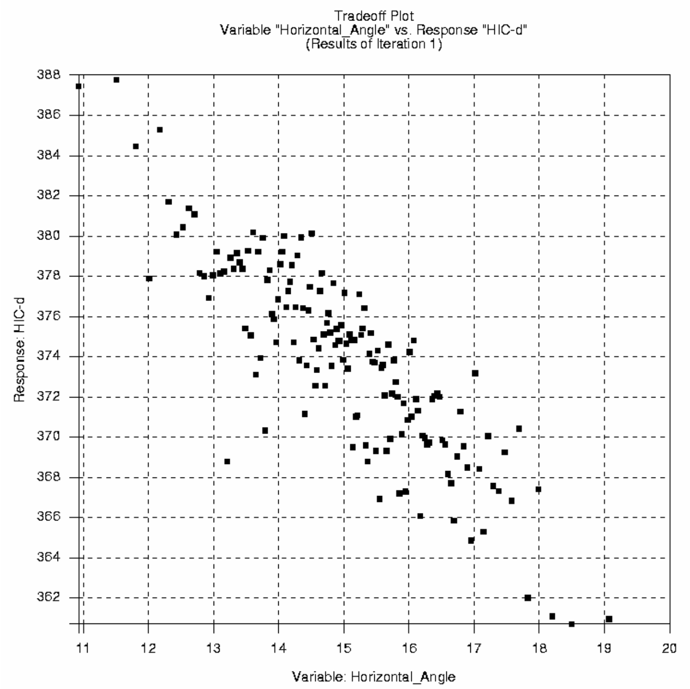


**Figure 6: HIC-d – Impact angle Monte Carlo simulation**

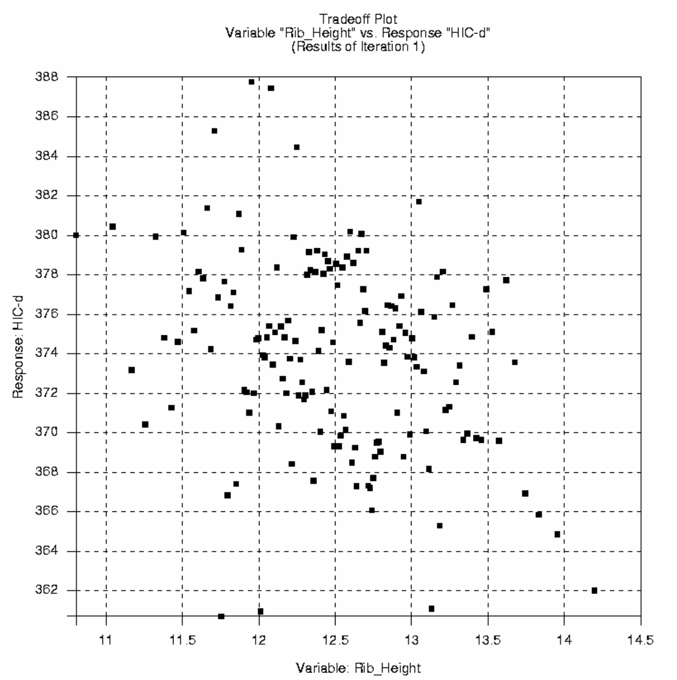


**Figure 7: HIC-d – Rib height Monte Carlo simulation**

The metamodel used is a quadratic response surface computed using a space filling experimental design of 60 points over a subregion two standard deviation wide around the design being investigated. The random variation is estimated to have a variance of 2.35 from the residuals. Results are computed from the metamodel using a Monte Carlo analysis of $10^6$ points.

The results for different probabilities of exceeding certain values are giving the following table. From the table the influence of the random variation on the probability of failure can again be seen.

| | **Monte Carlo 150 FE evaluations** | | **Metamodel 60 FE Evaluations [Value without random variation compensation]** | |
|---|---|---|---|---|
| | **Using Monte Carlo Values** | **Using Reliability Index** | **Monte Carlo** | **FORM** |
| Mean | 373.9 | | 373.9 | |
| Standard deviation | 4.85 | | 4.82 [4.21] | |
| P[HIC > 374] | 0.520 | 0.494 | 0.511 [0.517] | 0.514 [0.523] |
| P[HIC > 376] | 0.340 | 0.334 | 0.346 [0.328] | 0.349 [0.333] |
| P[HIC > 378] | 0.213 | 0.200 | 0.202 [0.167] | 0.202 [0.17] |
| P[HIC > 380] | 0.0733 | 0.105 | 0.0983 [0.0629] | 0.0960 [0.0649] |
| P[HIC > 382] | 0.0267 | 0.0479 | 0.0388 [0.0158] | 0.0351 [0.0166] |
| P[HIC > 384] | 0.0267 | 0.0189 | 0.0120 [0.0023] | 0.00922 [0.00243] |
| P[HIC > 386] | 0.0133 | 0.00638 | 0.00287 [0.00014] | 0.00159 [0.00016] |

### 4. Summary, Conclusions, and Recommendations

Predictable responses together with an estimate of their repeatability are desirable.

Probabilistic methods should have the following key properties:

- Establish link between cause and effect.
- Distinguish deterministic effects from random occurrences.
- Model and quantify uncertainty in the responses.

Using metamodels or the reliability index to compute reliability is preferred over a Monte Carlo analysis for reasons of cost and accuracy at small probabilities. Effective use of metamodels, especially for small probabilities, should incorporate (i) curvature and (ii) be accurate at the failure hyperplane.

The random variation – variation not associated with a change in the variables, though intrinsic to the structural event – can contribute to accuracy of reliability computations. The uncertainty can be used in conjunction with the metamodels when computing probabilities of events. However, if the random variation is not associated with the physics of the problem, then it should not be incorporated into the reliability computations and a metamodel filtering out the random variation should be used.

It is preferred to reduce random variation using careful modeling, consideration of the physics of the problem, and possibly redesign of the structure. Incorporating the noise component allows a better probabilistic quantification of results from the FEA analysis; it does not substitute for correct physics.

More advance meta-models may be considered; response surfaces are used here for clarity. The bias component of the residuals should be kept in mind.

## 5. Acknowledgements

We are indebted to everybody we talked to regarding this topic. In particular Paul du Bois provided us with an insightful summary of structural mechanics to expect during our investigation.

## 6. References

1. Robinson, D.G.; *A Survey of Probabilistic Methods Used In Reliability, Risk and Uncertainty Analysis: Analytical Techniques* I. SAND98-1189. June 1998.
2. Tu, J. and Choi, K.K. *Design potential concept for reliability-based design optimization.* Technical report R99-07. Center for computer aided design and department of mechanical engineering. College of engineering. University of Iowa. December 1999.
3. Thacker, B.H.; Riha, D.S.; Millwater, H.R.; and Enright, M.P. Errors and Uncertainties in Probabilistic Engineering Analysis. Proc. AIAA/ASME/ASCE/AHS/ASC 42nd Structures, Structural Dynamics, and Materials (SDM) Conf., AIAA 2001-1239, Seattle, WA, 16-19 April 2001.
4. Riha, D.S.; Thacker, B.H.; Hall, D.A.; Auel, T.R.; Pritchard, S.D. Capabilities and Applications of Probabilistic Methods in Finite Element Analysis. Presented at the Fifth ISSAT International Conference on Reliability and Quality in Design, Las Vegas, Nevada, August 11-13, 1999.
5. Box, G.E.P. and Draper, N.R., *Emperical Model-Building and Response Surfaces*, Wiley, New York, 1987.
6. *TrueGrid Manual*, XYZ Scientific Applications, Inc. Livermore, CA, 2000.